Notes for General Topology, Fall 2016

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Notation

X	set
f	function
$f : X \to Y$	function from X to Y
$f(A)$	image of A
$f^{-1}(B)$	inverse image of B
$\mathcal{P}(X)$	power set of X
$\mathbb{N}$	$\{1, 2, \dots\}$
$\mathbb{Q}$	the rational number
$\mathbb{R}$	the real number
$[a, b]$	real interval
$d(\cdot, \cdot)$	metric
(X, d)	metric space
$d_p(\cdot, \cdot)$	p-metric
$d_\infty(\cdot, \cdot)$	uniform metric
$\|\cdot\|_p$	p-norm
$\mathcal{B}[a, b]$	all bounded real-valued functions on $[a, b]$
$\mathcal{C}[a, b]$	all continuous real-valued functions on $[a, b]$
$B(x, \epsilon)$	open ball around x with radius $\epsilon > 0$
$\bar{B}(x, \epsilon)$	closed ball around x with radius $\epsilon > 0$
sup	least upper bound
inf	greatest lower bound
lim sup	limit superior
lim inf	limit inferior
$(X, \mathcal{T})$	topological space
$\mathcal{T}$	topology; collection of open sets
$\mathcal{T} = \sigma(\mathcal{B})$	topology generated by basis $\mathcal{B}$
$\mathcal{T} = \sigma(\mathcal{S})$	topology generated by subbasis $\mathcal{S}$
$\mathcal{C}$	the collection of closed set in $(X, \mathcal{T})$
A°	interior of A
$\bar{A}$	closure of A
A'	limit points of A
(x_n)	sequence
Λ	index set
$\prod_{\lambda \in \Lambda} X_\lambda$	Cartesian product of $\{X_\lambda\}_{\lambda \in \Lambda}$

1 Metric Spaces

DEFINITION 1.1 A metric d on a set X is a function from $X \times X$ to $\mathbb{R}$ that satisfies

a) $d(x, y) = d(y, x)$ for any $x, y \in X$ (Symmetry);

b) $d(x, y) \geq 0$ for any $x, y \in X$, with $d(x, y) = 0$ iff $x = y$ (Positive-definiteness);

c) $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$ (Triangle inequality).

A *metric space* is a pair (X, d) where X is a set and d is a metric on it.

- Metric space is just a straightforward generalization of the Euclidean space $\mathbb{R}^n$.
- The norm $\|\cdot\|_p : \mathbb{R}^n \to [0, \infty)$ given by $\mathbf{x} \mapsto \|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ defines the metric d_p with $d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p$. I shall sometimes use the two interchangeably to simplify notations.
- We prove Minkowski's inequality.

Theorem 1.2 (Minkowski's Inequality). *Let* $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$*, and let* $1 \leq p < \infty$*. Then*

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p. \tag{1}$$

◇

Proof. Let $\mathbf{x}_0 = (x_1^0, \ldots, x_n^0)$ and $\mathbf{y}_0 = (y_1^0, \ldots, y_n^0)$ be the unit vectors for $\mathbf{x}$ and $\mathbf{y}$ respectively, so that $\mathbf{x} = a\mathbf{x}_0$ and $\mathbf{y} = b\mathbf{y}_0$, where $a = \|\mathbf{x}\|_p$ and $b = \|\mathbf{y}\|_p$. Since the function $x \mapsto |x|^p$ is convex, for $t \in [0, 1]$ we have

$$\left|tx_i^0 + (1-t)y_i^0\right|^p \leq t\left|x_i^0\right|^p + (1-t)\left|y_i^0\right|^p \tag{2}$$

for each $i = 1, \ldots, n$. Summing over i, we have

$$\left\|t\mathbf{x}_0 + (1-t)\mathbf{y}_0\right\|_p^p \leq t + (1-t) = 1. \tag{3}$$

With $t = \frac{a}{a+b}$ the above becomes

$$\left\|\frac{a\mathbf{x}_0 + b\mathbf{y}_0}{a+b}\right\|_p^p \leq 1, \tag{4}$$

so that

$$\|\mathbf{x} + \mathbf{y}\|_p^p \leq (a+b)^p = (\|\mathbf{x}\|_p + \|\mathbf{y}\|_p)^p. \tag{5}$$

□

- The following proposition justifies our definition of the metric d_∞.

Proposition 1.3. *Let the metric* d_p *be defined by*

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for $1 \le p < \infty$*, and let the metric* d_∞ *be defined by*

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}.$$

Then as $p \to \infty$*,* $d_p(\mathbf{x}, \mathbf{y}) \to d_\infty(\mathbf{x}, \mathbf{y})$*.* ◆

Proof. We shall prove $\|\mathbf{x}\|_p \to \|\mathbf{x}\|_\infty$ as $p \to \infty$. Fix $\delta > 0$ and an $\mathbf{x} \in \mathbb{R}^n$, and let $S_\delta := \{1 \le i \le n : |x_i| \ge \|\mathbf{x}\|_\infty - \delta\}$. For example, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (2, 3, 5, 7, 6.9) \in \mathbb{R}^5$ and $\delta = 0.2$, then $\|\mathbf{x}\|_\infty = 7$, $\|\mathbf{x}\|_\infty - \delta = 7 - 0.2 = 6.8$, and since $7 > 6.8, 6.9 > 6.8$, we have $S_\delta = \{4, 5\}$, the index of the last two slots. We also use $|S_\delta|$ to denote the number of elements in S_δ, so $|S_\delta| = 2$ in the above example. Now,

$$\begin{aligned} \|\mathbf{x}\|_p &\ge \left(\sum_{S_\delta} (\|\mathbf{x}\|_\infty - \delta) \right)^{1/p} \\ &= \left(\|\mathbf{x}\|_\infty - \delta\right) |S_\delta|^{1/p}. \end{aligned}$$

Let $p \to \infty$, we have

$$\lim_{p\to\infty} \|\mathbf{x}\|_p \ge \|\mathbf{x}\|_\infty - \delta.$$

Since δ is arbitrary, we have

$$\lim_{p\to\infty} \|\mathbf{x}\|_p \ge \|\mathbf{x}\|_\infty. \tag{6}$$

On the other hand, $|x_i| \le \|\mathbf{x}\|_\infty$ for every $i = 1, 2, \dots, n$, so for $p > q$,

$$\begin{aligned} \|\mathbf{x}\|_p &= \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\ &= \left(\sum_{i=1}^{n} |x_i|^{p-q} |x_i|^q \right)^{1/p} \\ &\le \|\mathbf{x}\|_\infty^{(p-q)/p} \|\mathbf{x}\|_q^{q/p}. \end{aligned}$$

As $p \to \infty$, $(p-q)/p \to 1, q/p \to 0$, so we have

$$\lim_{p\to\infty} \|\mathbf{x}\|_p \le \|\mathbf{x}\|_\infty. \tag{7}$$

Combining Eq. (6) and Eq. (7) we see that

$$\lim_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty. \tag{8}$$

This proves that as $p \to \infty$, the metric d_p indeed converges to d_∞, since

$$\lim_{p\to\infty} d_p(\mathbf{x}, \mathbf{y}) = \lim_{p\to\infty} \|\mathbf{x} - \mathbf{y}\|_p = \|\mathbf{x} - \mathbf{y}\|_\infty = d_\infty(\mathbf{x}, \mathbf{y}).$$ □

DEFINITION 1.4 Let A be a subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is called *subadditive* if

$$f(a+b) \le f(a) + f(b)$$

for all $a, b \in A$.

Lemma 1.5. *If a function f is concave, and $f(0) \ge 0$, then f is subadditive.* ✻

Proof. Since f is concave, for $t \in [0,1]$, $f(tx) = f(tx + (1-t)\cdot 0) \ge tf(x) + (1-t)f(0) \ge tf(x)$. Thus

$$f(a) + f(b) = f\left((a+b)\frac{a}{a+b}\right) + f\left((a+b)\frac{b}{a+b}\right) \ge \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(a+b) = f(a+b).$$

□

Corollary 1.6. *Let (X,d) be a metric space. If $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 0$ is strictly increasing and concave, then $f \circ d$ is again a metric on X.* ♠

Proof. Symmetry and positive-definiteness for $f \circ d$ is straightforward. For the triangle inequality, since $d(x,y) \le d(x,z) + d(z,y)$ for the metric d, we have

$$f(d(x,y)) \le f(d(x,z) + d(z,y)) \le f(d(x,z)) + f(d(z,y)).$$

The first inequality holds since f is increasing on $[0,\infty)$, and the second inequality follows from the above lemma. □

- Let (X,d) be a metric space. Then $\sqrt{d}$ is another metric on X, since the function $f(x) = \sqrt{x}$ is concave and strictly increasing on $[0,\infty)$ with $f(0) = 0$.
- Let (X,d) be a metric space. Then

  $$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

  defines another metric on X, since $f(x) = \frac{x}{1+x}$ is increasing and concave on $[0,\infty)$. What's more, the metric $\tilde{d}$ is bounded, since $\lim_{x\to\infty} \frac{x}{1+x} = 1$. Later we will see that $\tilde{d}$ generates the same topology on X as d, so that the concept of "boundedness" is really just about metric, and has nothing to do with topology.
- By contrast, for a metric d on X, the function d^2 is no longer a metric on X. For example, let d be the Euclidean distance on the real line, and take $x = 0$, $y = 1$, and $z = 0.5$ on $\mathbb{R}$. Then $d^2(x,y) = 1$, but $d^2(x,z) + d^2(z,y) = 0.5^2 + 0.5^2 = 0.25 + 0.25 = 0.5$, so that the triangle inequality fails.
- Note that, for a set X, $d : X \times X \to \mathbb{R}$ given by $d(x,y) = 0$ for all $x, y \in X$ is not a metric, since positive-definiteness does not hold for d.

Let $\mathcal{B}[a,b]$ denote the space of bounded real-valued functions on $[a,b]$. Define a metric d on $\mathcal{B}[a,b]$ by

$$d(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|.$$

Then it is easy to see that d is a metric on $\mathcal{B}[a,b]$. It is called the *uniform metric* on $\mathcal{B}[a,b]$.

DEFINITION 1.7 Let (f_n) be sequence of real-valued functions defined on $A \subset \mathbb{R}$. We say that (f_n) *converges uniformly* to f if for every $\epsilon > 0$, there is $N > 0$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in A$.

The following proposition is immediate.

Proposition 1.8. *$f_n \to f$ uniformly if and only if $\lim_{n\to\infty} d(f_n, f) = 0$. In other words, $f_n \to f$ uniformly if and only if $(f_n) \subset \mathcal{B}[a,b]$ converges to $f \in \mathcal{B}[a,b]$ with respect to the uniform metric.* ◆

Proposition 1.9. *Let f_n be a sequence of continuous functions defined on $[a,b] \subset \mathbb{R}$. If $f_n \to f$ uniformly, then f is continuous.* ◆

Proof. Let $\epsilon > 0$. To prove f is continuous at a particular $x_0 \in [a,b]$, we need to find a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Now, according to triangle inequality,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

Each of the three terms on the right can be made small. Specifically, we can choose $n \geq N$ for some N such that $|f(x) - f_n(x)| < \epsilon/3$ and $|f_n(x_0) - f(x_0)| < \epsilon/3$, by uniform convergence of f_n to f. We can also choose some $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f_n(x) - f_n(x_0)| < \epsilon/3$, because each f_n is assumed to be continuous. Then

$$|f(x) - f(x_0)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the limiting function f is indeed continuous. □

Corollary 1.10. *Let $C[a,b]$ denote the space of continuous real-valued functions defined on $[a,b]$. Then $C[a,b]$ is closed in $\mathcal{B}[a,b]$.* ♠

Proof. $C[a,b]$ is closed in $\mathcal{B}[a,b]$ if and only if for every sequence (f_n) in $C[a,b]$ that converges to some $f \in \mathcal{B}[a,b]$, $f \in C[a,b]$. This is exactly what the above proposition says. □

- Another metric on $C[a,b]$ is given by

$$d(f,g) = \int_a^b |f(x) - g(x)| dx.$$

Proof. Symmetry is obvious. For positive definiteness, it suffices to prove that for $f \in C[a,b]$, $\int_a^b |f(x)|dx = 0$ implies $|f| = 0$ on $[a,b]$ (so that $f = 0$ on $[a,b]$). Suppose to the contrary, $|f(x_0)| \neq 0$ for some $x_0 \in [a,b]$, say $f(x_0) > 0$. Then since f is continuous on $[a,b]$, $f(y) > 0$ for all y sufficiently close to x_0. Indeed, choose $\epsilon > 0$ such that $f(x_0) - \epsilon > 0$. Then there exists $\delta > 0$ such that $|y - x_0| < \delta$ implies $|f(y) - f(x_0)| < \epsilon$, so that $0 < f(x_0) - \epsilon < f(y)$. Now

$$\int_a^b |f(x)|dx \geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|dx > 0,$$

contrary to our assumption that the integral on the left is zero. For triangle inequality, since $|f+g| \leq |f| + |g|$, we have

$$\int_a^b |f+g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g|.$$

□

The norm $\|\cdot\|_1 : C[a,b] \to \mathbb{R}, f \mapsto \int_a^b |f|$ gives rise to the metric d above. Similarly, for $1 \leq p < \infty$ we can define a norm $\|\cdot\|_p$ on $C[a,b]$ by

$$f \mapsto \|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}.$$

This can be seen as a generalization of the metric d_p on $\mathbb{R}^n$ to the function space $C[a,b]$. The proof of the Minkowski's inequality

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leq \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}$$

is basically same as Theorem 1.2, with integration in place of summation in Eq. (2). The crucial step is the convexity of the function $f(x) = x^p$ for $1 \leq p < \infty$.

1.1 Banach Fixed Point Theorem

DEFINITION 1.11 (CONTRACTION) Let (X,d) be a metric space. A mapping $T : X \to X$ is called a *contraction* on X if there is $\alpha \in (0,1)$ such that

$$d(Tx, Ty) \leq \alpha d(x,y)$$

for all $x, y \in X$.

Theorem 1.12 (Banach Fixed Point Theorem). *Let (X,d) be a nonempty complete metric space and suppose $T : X \to X$ is a contraction. Then there is a unique $x \in X$ such that $Tx = x$.* ◇

Proof.

1. First, for a contraction, its fixed point is necessarily unique. For suppose $x = Tx$ and $x' = Tx'$ are two fixed points of T. Then

$$d(x,x') = d(Tx,Tx') \leq \alpha d(x,x').$$

Since $\alpha < 1$, we have $d(x,x') = 0$ and thus by positive-definiteness of d we have $x = x'$.

2. Pick an arbitrary point $x_0 \in X$ and define a sequence (x_n) by

$$
\begin{aligned}
x_1 &= Tx_0;\\
x_2 &= Tx_1 = T^2x_0;\\
x_3 &= Tx_2 = T^3x_0;\\
&\vdots\\
x_n &= Tx_{n-1} = T^nx_0;\\
&\vdots
\end{aligned}
$$

From Definition 1.11, a contraction is continuous. If our (x_n) converges to some $x \in X$, then from $x_n = Tx_{n-1}$, we have

$$
x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx,
$$

so that $x \in X$ will be a fixed point of T.

3. We show (x_n) is Cauchy, thus converges to a point $x \in X$. Now,

$$
\begin{aligned}
d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1})\\
&\leq \alpha d(x_m, x_{m-1})\\
&= \alpha d(Tx_{m-1}, Tx_{m-2})\\
&\leq \alpha^2 d(x_{m-1}, x_{m-2})\\
&\vdots\\
&\leq \alpha^m d(x_1, x_0).
\end{aligned}
$$

Then by the triangle inequality, we have

$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)\\
&\leq (\alpha^m + \alpha^{m-1} + \cdots + \alpha^{n-1})d(x_1, x_0)\\
&= \frac{\alpha^m}{1-\alpha}(1-\alpha^{n-m})d(x_1, x_0)\\
&\leq \frac{\alpha^m}{1-\alpha}d(x_1, x_0) \to 0 \quad \text{as } m \to \infty.
\end{aligned}
$$

This proves (x_n) is Cauchy, and thus (x_n) converges by completeness of X. □

1.2 Characterization of Compact Metric Spaces

We have proved in class that a compact subset of a metric space is closed and bounded. The converse is true for $\mathbb{R}^n$ (see Corollary 9.13), but may not be true in general, the simplest example of which is the discrete metric on an infinite set X. Then the question is, what is the necessary and sufficient condition for a general metric space to be compact?

DEFINITION 1.13 (X,d) is *totally bounded* if given $\epsilon > 0$ (so that $X \subset \bigcup_{x\in X} B(x,\epsilon)$), there is some finite $x_1,\dots,x_n$ in X such that $X \subset \bigcup_{i=1}^{n} B(x_i,\epsilon)$.

Observation 1.14. Every compact metric space is totally bounded. ⊙

Theorem 1.15. *(X,d) is compact if and only if it is complete and totally bounded.* ◇

Proof. By Theorem 9.23 below, compactness and sequential compactness are equivalent for metric spaces. If (X,d) is sequentially compact, then it is obviously complete, since if a Cauchy sequence has a convergent subsequence, the sequence converges to the same limit. X, being compact, is also totally bounded.

Now suppose (X,d) is complete and totally bounded. Let (x_n) be a sequence in X. We shall find a convergent subsequence of (x_n). Note that any subset of X is also totally bounded, so we may apply totally boundedness to smaller and smaller subsets of X to capture a Cauchy subsequence.

First, finitely many $B(x,1), x \in X$ cover X, so a single $B(1)$ must capture infinitely many items of (x_n). Pick $x_{n_1} \in B(1)$. Finitely many $B(x,1/2), x \in B(1)$ cover $B(1)$, so infinitely many items of $\{x_n\} \cap B(1)$ must fall into at least one such $B(1/2)$. Pick $x_{n_2} \in B(1/2) \cap B(1)$. Similarly, we can pick $x_{n_3} \in B(1/3) \cap B(1/2)$......Continuing this way, we obtain a subsequence (x_{n_k}) such that

$$x_{n_k} \in B(\frac{1}{N}) \quad \text{whenever } k \geq N.$$

The subsequence (x_{n_k}) is thus Cauchy, for

$$d(x_{n_k}, x_{n_l}) \leq \frac{1}{N} \quad \text{whenever } k, l \geq N.$$

Since X is assumed to be complete, (x_{n_k}) converges. This proves every sequence in X has a convergent subsequence. □

2 Sets

- Two sets A and B are equal ($A = B$) if and only if $A \subseteq B$ and $B \subseteq A$. This simple fact is used extensively in our proof of various theorems and propositions in topology. For example, if we want to prove $G = U \cap Y$, then we may prove $G \subseteq (U \cap Y)$ as well as $(U \cap Y) \subseteq G$.
- $A \subseteq B$ means for *every* $x \in A$, x is also in B.
- $A \cup B = \{x : x \in A \text{ or } x \in B\}$; $\quad A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- If for a family of sets $\{U_i\}_{i\in I}$, $U_i \subseteq X$ for each index $i \in I$, then their union is also a subset of X, namely $\bigcup_{i\in I} U_i \subseteq X$.
- De Morgan's Law:

$$(A \cup B)^c = A^c \cap B^c; \tag{9}$$

$$(A \cap B)^c = A^c \cup B^c. \tag{10}$$

Or write $X \setminus A$ for A^c if we were to make the ambient space explicit:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B); \tag{11}$$

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B). \tag{12}$$

More generally, for an arbitrary collection of subsets $\{U_i\}_{i \in I}$ in X, we have

$$X \setminus \left(\bigcup_{i \in I} U_i \right) = \bigcap_{i \in I} \left(X \setminus U_i \right); \tag{13}$$

$$X \setminus \left(\bigcap_{i \in I} U_i \right) = \bigcup_{i \in I} \left(X \setminus U_i \right). \tag{14}$$

We will also use De Morgan's law extensively.

- **Lemma 2.1.** $C \setminus (B \setminus A) = (C \setminus B) \cup (C \cap A)$. ✻

Proof. Just write out the definitions:

$$B \setminus A = \{x \in B \text{ and } x \notin A\},$$

so

$$\begin{aligned} C \setminus (B \setminus A) &= \{x \in C : x \notin B \text{ or } x \in A\} \\ &= \{x \in C : x \notin B\} \cup \{x \in C : x \in A\} \\ &= (C \setminus B) \cup (C \cap A). \end{aligned}$$

□

- **Lemma 2.2.** $(B \setminus A) \cap C = (B \cap C) \setminus A = B \cap (C \setminus A)$. ✻

Proof.

$$\begin{aligned} (B \setminus A) \cap C &= \{x \in B \text{ and } x \notin A \text{ and } x \in C\} \\ &= \{x \in B \text{ and } x \in C \text{ and } x \notin A\} \\ &= \{x \in B \cap C \text{ and } x \notin A\} \\ &= (B \cap C) \setminus A. \end{aligned}$$

We can also write

$$\begin{aligned} (B \setminus A) \cap C &= \{x \in B \text{ and } x \notin A \text{ and } x \in C\} \\ &= \{x \in B \text{ and } x \in C \text{ and } x \notin A\} \\ &= \{x \in B \text{ and } x \in C \setminus A\} \\ &= B \cap (C \setminus A). \end{aligned}$$

□

- Given two propositions p and q, $p \Rightarrow q$ means "p implies q", namely, p being true is sufficient for q being true, or q being true is necessary for the truth of p (if q is not true, then p can not be true either, since if it was, then we can deduce that q is true, a contradiction). Thus p being true is a sufficient condition for q being true, and q being true is a necessary condition for p being true.

- $p \Leftrightarrow q$ means p being true is necessary and sufficient for q being true, namely, p holds *if and only if* q holds.

- We use "$p \vee q$" to denote "p or q". $p \vee q$ is true if and only if at least one of them is true.

- We use "$p \wedge q$" to denote "p and q". $p \wedge q$ is true if and only if both of them is true.

- We use "$\neg p$" to denote the negation of p. Recall from high school math that $p \Rightarrow q$ if and only if $\neg q \Rightarrow \neg p$.

- De Morgan's law: $\neg(p \vee q) = (\neg p) \wedge (\neg q)$ and $\neg(p \wedge q) = (\neg p) \vee (\neg q)$.

- Given two sets X and Y, a *function* from X to Y associates each $x \in X$ with an element $f(x) \in Y$. For a subset U of Y, its *preimage*, or *inverse image*, is the set $f^{-1}(U) = \{x \in X : f(x) \in U\}$, which is a subset of X. It is easy to see that for $U, V \subseteq Y$, we have

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A);$$

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V);$$
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V).$$

More generally,

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} V_i\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda); \tag{15}$$

$$f^{-1}\left(\bigcap_{\lambda \in \Lambda} V_i\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(V). \tag{16}$$

  - For more on set theory, see my notes here.

3 Upper and Lower Bounds

We have learned least upper bound, greatest lower bound, lim inf and lim sup. It's important to remember and understand the definitions of them clearly.

DEFINITION 3.1 For $A \subset \mathbb{R}$, if there is a number M such that $a \leq M$ for all $a \in A$, then M is called an *upper bound* for A. Similarly, if there is a number l such that $l \leq a$ for all $a \in A$, then l is called an *lower bound* for A.

DEFINITION 3.2 A *least upper bound* for $A \subset \mathbb{R}$ is a number x such that

- x is an upper bound for A;
- If $r < x$, then r is not an upper bound for A.

We write $x = \sup A$.

Consider $A = \{q \in \mathbb{Q} \mid q^2 < 2\}$ as a subset of $\mathbb{Q}$. A does not have a least upper bound in $\mathbb{Q}$. Every rational number in $[\sqrt{2}, \infty)$ is an upper bound for A, but no matter how close to $\sqrt{2}$ our choice of $q \in \mathbb{Q} \cap [\sqrt{2}, \infty)$ is, we can always find a rational number q' such that $\sqrt{2} < q' < q$, who is even closer to $\sqrt{2}$ and hence to A. Thus, the set A does not have a least upper bound in $\mathbb{Q}$. $\sqrt{2}$ is a "gap" that makes the rational number "incomplete". To fill the gaps, one constructs real numbers $\mathbb{R}$ from rational numbers (via Dedekind cut), and our real numbers would be complete, in the sense that:

Theorem 3.3. *Every subset of $\mathbb{R}$ that is bounded above has a least upper bound.* ◇

Placing our set $A = \{q \in \mathbb{Q} \mid q^2 < 2\}$ in $\mathbb{R}$, we see that $\sqrt{2}$ is the least upper bound for A.

Now consider Definition 3.2. Note that given $\epsilon > 0$, since $x - \epsilon < x$, it is not an upper bound for A, so that there exists some $a \in A$ such that $a > x - \epsilon$. Otherwise, if no such $a \in A$ exists, then $a \leq x - \epsilon$ for all $a \in A$, so that $x - \epsilon$ would be an upper bound for A, a contradiction. Thus we have

Proposition 3.4. $x = \sup A$ *if and only if* $a \leq x$ *for all* $a \in A$, *and for every* $\epsilon > 0$, *there is* $a \in A$ *such that* $a > x - \epsilon$. ◆

DEFINITION 3.5 A *greatest lower bound* for $A \subset \mathbb{R}$ is a number x such that

- x is a lower bound for A;
- If $r > x$, then r is not a lower bound for A.

We write $x = \inf A$.

Proposition 3.6. $x = \inf A$ *if and only if* $x \leq a$ *for all* $a \in A$, *and for every* $\epsilon > 0$, *there is* $a \in A$ *such that* $a < x + \epsilon$. ◆

There are three equivalent definitions of lim sup and lim inf. Be sure to remember and understand all of them.

DEFINITION 3.7 For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, we write $\limsup_{n\to\infty} a_n = L$ if there is a number L such that

- For any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $a_n < L + \epsilon$ for all $n \geq N$;
- For any $\epsilon > 0$ and $N \in \mathbb{N}$, there is $n \geq N$ such that $a_n > L - \epsilon$.

This means that all but finitely many a_n lie to the left of $L + \epsilon$, while infinitely many a_n lie to the right of $L - \epsilon$. Also, the number L that satisfies the above conditions is unique.

For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, we write $\liminf_{n\to\infty} a_n = l$ if there is a number l such that

- For any $\epsilon > 0$ and $N \in \mathbb{N}$, there is $n \geq N$ such that $a_n < l + \epsilon$;
- For any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $a_n > l - \epsilon$ for all $n \geq N$.

This means that all but finitely many a_n lie to the right of $l - \epsilon$, while infinitely many a_n lie to the left of $l + \epsilon$. The number l that satisfies the above conditions is unique.

DEFINITION 3.8 For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, let

$$E = \{x \mid x = \lim_{k\to\infty} a_{n_k} \text{ for some subsequence } (a_{n_k}) \text{ of } (a_n)\}.$$

E is the set of all subsequencial limits of (a_n). We define

$$\limsup_{n\to\infty} a_n := \sup E; \qquad \liminf_{n\to\infty} a_n := \inf E.$$

DEFINITION 3.9 For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, we define

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right),$$

and similarly,

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right).$$

Sometimes we omit "$n \to \infty$" in the bottom of lim sup and lim inf to simplify notations. $(\sup_{k\geq n} a_k)$ in Definition 3.9 is an abbreviation for $\sup\{a_k : k \geq n\}$, and $(\inf_{k\geq n} a_k)$ is an abbreviation for $\inf\{a_k : k \geq n\}$. Note that $u_n := \sup\{a_k : k \geq n\}$ is a decreasing sequence, so $\lim_{n\to\infty} u_n$ converges in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. Similarly, $\inf\{a_k : k \geq n\}$ is an increasing sequence, so it converges as well.

Proposition 3.10. *Definition 3.7 and Definition 3.8 are equivalent.* ♦

Proof. We prove the case for lim sup. The case for lim inf is similar. Let $x = \sup E$, where E is as in Definition 3.8. We show x satisfies the two conditions in Definition 3.7. First, is $a_n < x + \epsilon$ eventually? This is true, for otherwise, we would have $a_n \geq x + \epsilon$ infinitely often, which implies that $a_{n_k} \geq x + \epsilon$ infinitely often for some subsequence (a_{n_k}). But then $x \geq \lim_{k\to\infty} a_{n_k} \geq x + \epsilon$, which is not true. So x do satisfies the first condition.

Is $a_n > x - \epsilon$ infinitely often? This can also be easily seen to be true: since $x - \epsilon$ is not an upper bound for E, there is $\xi \in E$ such that $\xi > x - \epsilon$. Since ξ is a limit of some subsequence of (a_n), we have that subsequence $> x - \epsilon$ eventually, so our desired conclusion holds. □

Proposition 3.11. *Definition 3.7 and Definition 3.9 are equivalent.* ◆

Proof. We prove the case for limit superior. Let $u_n = \sup_{k\geq n} a_k$, and let $u = \lim_{n\to\infty} u_n$. We show that u satisfies the two conditions in Definition 3.7. Now, the limit means that

$$\text{For every } \epsilon > 0\text{, there is } N \in \mathbb{N}\text{, such that } |u_n - u| < \epsilon \text{ for all } n \geq N.$$

Thus given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $a_n \leq u_n < u + \epsilon$ for all $n \geq N$. This proves the first condition.

On the other hand, invoking the definition of least upper bound, we see that there is an integer $m > n \geq N$, such that $a_m > u_n - \epsilon$. Since $u_n > u - \epsilon$, we have $a_n > u_n - \epsilon > (u - \epsilon) - \epsilon = u - 2\epsilon$. This shows that u also satisfies the second condition of Definition 3.7. This proves the equivalence of the two definitions. □

Proposition 3.12. *Let* $\limsup a_n = L$*. Then there is a subsequence* a_{n_k} *of* (a_n) *such that*

$$\lim_{k\to\infty} a_{n_k} = L.$$

◆

Proof. Resort to Definition 3.7, there exists a_{n_1} such that $|a_{n_1} - L| < \frac{1}{2}$. Similarly, there exists a_{n_2} such that $|a_{n_2} - L| < \frac{1}{2^2}$. Continuing this way, we have a subsequence (a_{n_k}) such that $|a_{n_k} - L| < \frac{1}{2^k}$. Then it is clear that $\lim_{k\to\infty} a_{n_k} = L$. □

In the notation of Definition 3.8, the proposition above says that $\sup E \in E$.

4 Definition and Examples

Definition 4.1 A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

a) $\varnothing, X \in \mathcal{T}$;

b) arbitrary union of elements of $\mathcal{T}$ is in $\mathcal{T}$;

c) finite intersections of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Elements of $\mathcal{T}$ is called *open sets*. The pair $(X, \mathcal{T})$ is called a *topological space*.

4.1 Examples of Topological Spaces

- Let $X = \{a, b, c\}$. Then $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ is a topology on X, as you can easily verify.
- For a set X, $\mathcal{T} = \{\emptyset, X\}$ is a topology on X, called the *trivial topology* on X.
- For a set X, the power set $\mathcal{P}(X)$, the set that consists of *all* subsets of X, is a topology on X. It is called the *discrete topology* on X.
- It is immediate that a metric space (X, d) is a topological space. The topology on X is the collection of all open sets. Denote this topology by $\mathcal{O}(d)$. On the other hand, given a topology $\mathcal{T}$ on a set X, does there exist a metric d on X that generates the topology $\mathcal{T}$?

DEFINITION 4.2 A topological space $(X, \mathcal{T})$ is called *metrizable* if there exists some metric d on X such that $\mathcal{T} = \mathcal{O}(d)$.

Metric space is something we are familiar with. Given a topological space, we may want to determine whether it is metrizable, i.e., whether it is some metric space. So the question is, under what conditions is a topological space metrizable? You can have this question as a motivation for studying general topology. In particular, for every new concepts and definitions we are going to learn, think about how they are abstracted from metric space, and conversely, whether those concepts are enough to characterize certain or all metric spaces.

5 Basis

Sometimes we want to build a topology on a set X from something that is familar to us. Or conversely, given a topology on X, which may be too large to describe, we may wish to describe it in terms of something smaller. This leads to the concept of *basis*.

DEFINITION 5.1 Let X be a set. A *basis* $\mathcal{B}$ on X is a collection of subsets of X such that

a) For each $x \in X$, there is some $B \in \mathcal{B}$ such that $x \in B$;

b) For each $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Recall the definition of open set in metric space: a set U is open if for every $x \in U$ we can find some open ball B such that $x \in B \subset U$. Given a basis $\mathcal{B}$ of X, we would like to model the situation for metric space to generate a topology $\mathcal{T}$ on X from $\mathcal{B}$. So we have the following construction. We let

$$\mathcal{T} = \{\, U \subset X \mid \text{for every } x \in U \text{ there is some } B \in \mathcal{B} \text{ such that } x \in B \subset U \,\}.$$

Proposition 5.2. *$\mathcal{T}$ is indeed a topology on X.* ♦

Proof. We verify $\mathcal{T}$ satisfies the three conditions in Definition 4.1.

a) $\varnothing, X \in \mathcal{T}$.

b) $\{U_\lambda\} \subset \mathcal{T}$ implies $U = \bigcup U_\lambda \in \mathcal{T}$. Indeed, let $x \in U$. Then $x \in U_\lambda$ for some particular $U_\lambda \in \mathcal{T}$. Then by definition there is some $B \in \mathcal{B}$ such that $x \in B \subset U_\lambda$. Then $x \in B \subset U$, so that $U \in \mathcal{T}$ as well.

c) Let $U_1, U_2 \in \mathcal{T}$. We want $U_1 \cap U_2 \in \mathcal{T}$. Let $x \in U_1 \cap U_2$. Then since $x \in U_1$ as well as $x \in U_2$, there are some B_1 and B_2 such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Since $\mathcal{B}$ is a basis, we have $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$ for some $B_3 \in \mathcal{B}$. This shows that $U_1 \cap U_2 \in \mathcal{T}$. The case for finite intersection follows by induction. □

Compare the definition of basis with that of topology, we find that the former is easier. For example, it is easy to see that the collection of open balls in a metric space (X, d) is a basis on X. Given a basis on X, we can construct a topology on X and speak of "open sets". The following proposition further explains in what sense $\mathcal{B}$ would be a basis for a topology $\mathcal{T}$.

Proposition 5.3. *Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$. Then every element in $\mathcal{T}$ is a union of elements in $\mathcal{B}$.* ♦

Proof. Recall in Problem Set 1, we have learned that in a metric space, an open set can be expressed as a union of open balls. Namely, for $U \subset X$, we have

$$U = \bigcup_{x \in U} B(x, \epsilon_x).$$

The situation here is just a model of this. Let $U \in \mathcal{T}$. Then for every $x \in U$ there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then

$$U = \bigcup_{x \in U} B_x.$$

□

Notation. We write $\mathcal{T} = \sigma(\mathcal{B})$ if $\mathcal{T}$ is generated by the basis $\mathcal{B}$.

DEFINITION 5.4 Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$, then we say $\mathcal{T}'$ is *finer* than $\mathcal{T}$, and $\mathcal{T}$ is *coarser* than $\mathcal{T}'$. $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$ (or $\mathcal{T}$ is *strictly coarser* than $\mathcal{T}'$) if the inclusion is proper. If neither topology includes the other, then we say they are not comparable.

Basis makes it easier to compare topologies.

Lemma 5.5. *Let $\mathcal{T} = \sigma(\mathcal{B})$ and $\mathcal{T}' = \sigma(\mathcal{B}')$. The following are equivalent:*

(1) $\mathcal{T} \subset \mathcal{T}'$;

(2) For every $x \in X$ and every $x \in B \in \mathcal{B}$, there is B' such that $x \in B' \subset B$. ✻

Proof. (2) ⇒ (1): Let $U \in \mathcal{T} = \sigma(\mathcal{B})$. Then given $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$. By the assumption of (2), we have $x \in B' \subset B \subset U$ for some $B' \in \mathcal{B}'$. Then $U \in \mathcal{T}' = \sigma(\mathcal{B}')$.

(1) ⇒ (2): Note that, given $B \in \mathcal{B}$, since $B \in \mathcal{T}$, we have $B \in \mathcal{T}'$. Then (2) holds by the definition of $\mathcal{T}'$. □

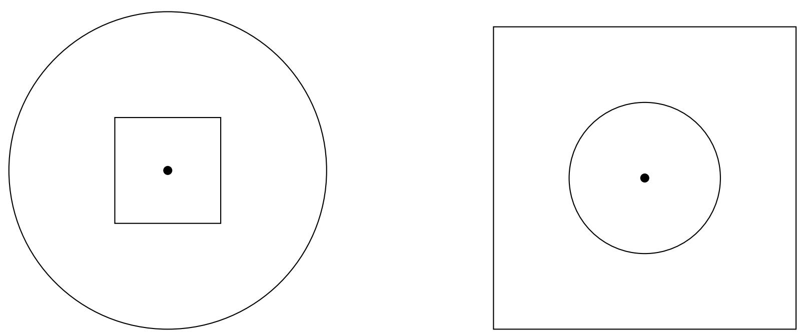
Thus, to compare two topologies on a set X, we only need to compare their basis.

5.1 Examples

- We have mentioned that, for a metric space (X, d), the collection of open balls is a basis. Thus, for example, the collection of open intervals (a, b) in $\mathbb{R}$ is a basis for $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. Similarly, an element of a basis for $(\mathbb{R}^2, d_2)$ is an open disk, where $d_2(x, y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}$ for $x = (x_1, x_2), y = (y_1, y_2)$. Another metric on $\mathbb{R}^2$ is $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Although the two metrics are different (two points in $\mathbb{R}^2$ have different "distances" under the two metrics), they generate the same topology on $\mathbb{R}^2$.

  Proposition 5.6. $\mathcal{O}(d_2) = \mathcal{O}(d_\infty)$. ◆

  Proof. A drawing would be clear enough to illustrate Lemma 5.5.

  □

  Similarly, all of the metrics d_p for $1 \leq p < \infty$ generate the same topology on $\mathbb{R}^n$. Thus, we see that topology concerns only *open sets*, and would forget the geometric property of "distance".

- For $(\mathbb{R}^n, d_p)$, and its collection of its open balls $\mathcal{B}$, $\mathcal{T} = \sigma(\mathcal{B})$ can be generated by a much smaller basis: the collection of open balls with rational radius $\mathcal{B}_\mathbb{Q}$, which is countable. Every open ball with real radius inscribes a smaller open ball with rational radius, as it can also be inscribed by a larger one. This shows that a metric space can have a *countable* basis.

- Let's consider a different basis on $\mathbb{R}$. We let $\mathcal{B}'$ be the collection of all half-open intervals of the form

  $$[a, b) = \{x : a \leq x < b\}.$$

  Is the topology generated by this $\mathcal{B}'$ the same as the usual one?

  Proposition 5.7. *$\mathcal{T}' = \sigma(\mathcal{B}')$ is strictly finer than the usual topology on $\mathbb{R}$.* ◆

Proof. We still apply Lemma 5.5. For every $x \in (a, b)$, $x \in [x, b) \subset (a, b)$, and $[x, b) \in \mathcal{B}'$. On the other hand, given a basis element $[y, d)$ in $\mathcal{B}'$, there is no interval (a, b) around y such that $y \in (a, b) \subset [y, d)$. □

Notation. We denote $(\mathbb{R}, \mathcal{T}')$ by $\mathbb{R}_\ell$, and call $\mathcal{T}'$ the *lower limit topology* on $\mathbb{R}$.

6 Closed Sets, Limit Points, Convergence of Sequences, Hausdorff Spaces

Our definition of closed sets is the same as in the case for metric spaces.

DEFINITION 6.1 Let $(X, \mathcal{T})$ be a topological space. $A \subset X$ is called *closed* if $X \setminus A \in \mathcal{T}$.

Proposition 6.2. *Let $\mathcal{C}$ denote the set of all closed set in $(X, \mathcal{T})$. Then*

a) $\varnothing, X \in \mathcal{C}$;

b) $\{A_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{C} \Rightarrow \bigcap_{\lambda \in \Lambda} A_\lambda \in \mathcal{C}$;

c) $\{A_1, A_2, \ldots, A_n\} \subset \mathcal{C} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{C}$. ◆

Proof.

$$X \setminus \left(\bigcap A_\lambda\right) = \bigcup \left(X \setminus A_\lambda\right) \in \mathcal{T}$$

and

$$X \setminus \left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} \left(X \setminus A_i\right) \in \mathcal{T}.$$

□

Exercise 6.3. If $U \in \mathcal{T}$, $C \in \mathcal{C}$, then $U \setminus C \in \mathcal{T}$, and $C \setminus U \in \mathcal{C}$. ◴

Proof. By Lemma 2.1, $X \setminus (U \setminus C) = (X \setminus U) \cup (X \cap C) = (X \setminus U) \cup C \in \mathcal{C}$, and $X \setminus (C \setminus U) = (X \setminus C) \cup (X \cap U) = (X \setminus C) \cup U \in \mathcal{T}$. □

The interior and closure of a set are defined in the same way as for metric spaces.

DEFINITION 6.4 Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. Then

1. The *interior* of A is defined to be $A^\circ = \bigcup\{G : G \subset A, G \in \mathcal{T}\}$.
2. The *closure* of A is defined to be $\bar{A} = \bigcap\{F : A \subset F, F \in \mathcal{C}\}$.

The interior of A is the biggest open set contained in A, and the closure of A is the smallest closed set containing A. A is open iff $A = A^\circ$, and A is closed iff $A = \bar{A}$.

Terminology. If a point x in X is in some open set U, then we call U a *neighborhood* of x.

The definition of limit point is a direct generalization from metric spaces.

DEFINITION 6.5 x is called a *limit point* of A if for every neighborhood U of x, there is some $y \in A \cap U$ such that $y \neq x$. We denote the set of all limit points of A by A'.

Some authors define the closure of a set A to be $A \cup A'$. These two definitions are equivalent.

Proposition 6.6. $\bar{A} = \bigcap\{F : A \subset F, F \in C\} = A \cup A'$. ◆

Proof. We first prove $x \in \bar{A}$ if and only if for every neighborhood U of x, $A \cap U \neq \varnothing$. We prove the contrapositive: $x \notin \bar{A}$ if and only if there is some neighborhood U of x such that $U \cap A = \varnothing$. Now

1. $x \notin \bar{A} \Rightarrow x \in X \setminus \bar{A}$, so that if we let $U = X \setminus \bar{A}$, then $U \cap A = \varnothing$.
2. $x \in U,\ U \cap A = \varnothing \Rightarrow A \subset X \setminus U \Rightarrow \bar{A} \subset X \setminus U \Rightarrow x \notin \bar{A}$.

Return to our proposition ,we first have $\bar{A} \subset A \cup A'$: let $x \in \bar{A}$. If $x \in A$, we are done. If $x \notin A$, then for every neighborhood U of x, $A \cap U \neq \varnothing$. Any element in $A \cap U$ cannot be x, so x is a limit point of A. Conversely, by what we have just proved, it is obvious that $A \cup A' \subset \bar{A}$. □

Corollary 6.7. *A is closed if and only if $A' \subset A$.* ♠

Some properties of closure:

Exercise 6.8. Let A, B, and $\{A_\lambda\}$ be subsets of $(X, \mathcal{T})$. Prove the following:

(a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(c) $\bigcup \bar{A}_\lambda \subset \overline{\bigcup A_\lambda}$. ◴

Proof. (a) From our proof of Proposition 6.6, $x \in \bar{A}$ if and only if every neighborhood of x has nonempty intersection with A. Since $A \subset B$, it is also true that every neighborhood of x has nonempty intersection with B. Thus $x \in \bar{B}$.

(b) First, since $A \subset \bar{A}$ and $B \subset \bar{B}$, $A \cup B \subset \bar{A} \cup \bar{B}$, which is closed. Since the closure of $A \cup B$ is the smallest closed set containing $A \cup B$, we have $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. On the other hand, $A \subset A \cup B$, so that $\bar{A} \subset \overline{A \cup B}$ by (a). Similarly, $\bar{B} \subset \overline{A \cup B}$. Thus $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

(c) Since $A_\lambda \subset \bigcup A_\lambda$ for each λ, $\bar{A}_\lambda \subset \overline{\bigcup A_\lambda}$ for each λ, by (a). Then $\bigcup \bar{A}_\lambda \subset \overline{\bigcup A_\lambda}$.
Although it is true that $A_\lambda \subset \bar{A}_\lambda$ for each λ and thus $\bigcup A_\lambda \subset \bigcup \bar{A}_\lambda$, $\bigcup \bar{A}_\lambda$ may not be closed, since arbitrary union of close sets need not be closed. Thus the other inclusion does not hold. An example would be $A_n = \bar{A}_n = \{\frac{1}{n}\}, n = 1, 2, \ldots$. In this case, $\bigcup \bar{A}_n = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$, $\bigcup A_n = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$, but $\overline{\bigcup A_n} = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$. □

Exercise 6.9. Let A, B, and $\{A_\lambda\}$ be subsets of $(X, \mathcal{T})$. Prove the following:

(a) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

(b) $\overline{\bigcap A_\lambda} \subset \bigcap \bar{A}_\lambda$.

(c) $\bar{A} \setminus \bar{B} \subset \overline{A \setminus B}$. ◴

Proof. (a) $A \cap B \subset A$ and $A \cap B \subset B$, so by (a) $\overline{A \cap B} \subset \bar{A}$ and $\overline{A \cap B} \subset \bar{B}$. Thus $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. The other inclusion need not hold. For example, let $A = (0, \frac{1}{2})$ and $B = (\frac{1}{2}, 1)$ be two open intervals in $\mathbb{R}$. Then $A \cap B = \varnothing$, so that $\overline{A \cap B} = \bar{\varnothing} = \varnothing$. On the other hand, $\bar{A} = [0, \frac{1}{2}]$ and $\bar{B} = [\frac{1}{2}, 1]$ so that $\bar{A} \cap \bar{B} = \{\frac{1}{2}\}$.

(b) Since $\bigcap A_\lambda \subset A_\lambda$ for each λ, $\overline{\bigcap A_\lambda} \subset \bar{A}_\lambda$ for each λ, by (a). Then $\overline{\bigcap A_\lambda} \subset \bigcap \bar{A}_\lambda$. Again, the inverse inclusion needs not hold.

(c) Since $B \subset \bar{B}$, we have $\bar{A} \setminus \bar{B} \subset \bar{A} \setminus B$. Let $x \in \bar{A} \setminus \bar{B}$, so that $x \in \bar{A}$ and $x \notin B$. For every neighborhood U of x, $A \cap U \neq \varnothing$. Now by Lemma 2.2,

$$(A \setminus B) \cap U = (A \cap U) \setminus B.$$

We know $A \cap U \neq \varnothing$. But $(A \cap U) \setminus B \neq \varnothing$ either, since $A \cap U$ can not be a subset of B (we have at least $x \notin B$). This shows that $x \in \overline{A \setminus B}$ by our proof of Proposition 6.6. □

DEFINITION 6.10 Let $(X, \mathcal{T})$ be a topological space. A sequence of point (x_n) in X is said to *converge* to $x \in X$ if every neighborhood of x contains all but finitely many points of (x_n).

If a topology has too few open sets, it may happen that a sequence may converge to more than one point. In the extreme case of trivial topology, every sequence converges to every point in the space! Thus, the coarser a topology is, the easier it is for a sequence to converge; the finer a topology is, the more difficult it is for a sequence to converge.

DEFINITION 6.11 $(X, \mathcal{T})$ is called a *Hausdorff space* if for every $x \neq y \in X$, there are $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Namely, in a Hausdorff space, every pair of distinct points can be separated by open sets.

- Any metric space (X, d) is Hausdorff. Indeed, let $x \neq y \in X$. Then $d(x, y) > 0$, so if we let $r = \frac{1}{3} d(x, y)$, we then have $B(x, r) \cap B(y, r) = \varnothing$.

Observation 6.12. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies defined on X such that $\mathcal{T} \subset \mathcal{T}'$. If $(X, \mathcal{T})$ is Hausdorff, then $(X, \mathcal{T}')$ is Hausdorff. ⊙

Proposition 6.13. *Let $(X, \mathcal{T})$ be a Hausdorff space. Then every sequence in X converges to at most one point.* ◆

Proof. Suppose $x_n \to x$. Then for any $y \neq x$, we can find some $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. $x_n \in U$ for all but finitely many n, so that V can not. □

Exercise 6.14. Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. ◴

Proof. $U \cap V = \emptyset$ if and only if $(U \times V) \cap \Delta = \emptyset$. □

Exercise 6.15. Every finite point set in a Hausdorff space is closed. ◴

Proof. It suffices to prove each one point set $\{x_0\}$ is closed. But $X \setminus \{x_0\}$ is open, since for every $y \in X \setminus \{x_0\}$ we can find $U, V \in \mathcal{T}$ such that $y \in V \subset X \setminus U \subset X \setminus \{x_0\}$. □

DEFINITION 6.16 A topological space $(X, \mathcal{T})$ in which finite point sets are closed is called a T_1 *space*.

Exercise 6.17. Let $(X, \mathcal{T})$ be a T_1 space, and let x be a limit point of $A \subset X$. Then every neighborhood of x contains infinitely many points of A. ◴

Proof. Suppose for some $x \in U \in \mathcal{T}$, $(U \setminus \{x\}) \cap A = \{x_1, \dots, x_n\} \in \mathcal{C}$. By Exercise 6.3, $U \setminus \{x_1, \dots, x_n\} \in \mathcal{T}$, and so it is a neighborhood of x not containing any point of A, a contradiction. □

Exercise 6.18. $(X, \mathcal{T})$ is a T_1 space if and only if for every pair of points each has a neighborhood not containing the other. ◴

Proof. "⇒": Let $x \neq y$. Then $X \setminus \{x\}$ is the open neighborhood of y not containing x, and $X \setminus \{y\}$ is the open neighborhood of x not containing y.

"⇐": Given $x_0 \in X$, we prove $X \setminus \{x_0\}$ is open. Let $y \in X \setminus \{x_0\}$. Then by assumption there is $y \in V$ such that $V \cap \{x_0\} = \emptyset$. Then $y \in V \subset X \setminus \{x_0\}$. □

7 Continuous Functions, Subbasis, Subspace Topology, Product Topology

DEFINITION 7.1 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is said to be *continuous* if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{T}_Y$.

DEFINITION 7.2 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. If there is a bijective function $f : X \to Y$ such that

(1) $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{T}_Y$,

(2) $f(U) \in \mathcal{T}_Y$ for every $U \in \mathcal{T}_X$,

then X and Y are called *homeomorphic*. f is called a *homeomorphism* between X and Y.

Exercise 7.3. If $f : X \to Y$ is continuous, then for each $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y. ◴

Proof. Let $f(x) \in V$, where V is open. Then $x \in f^{-1}(V)$, which is also open. Thus $x_n \in f^{-1}(V)$ for all but finitely many n. Then $f(x_n) \in V$ for all but finitely many n. This proves $\lim_{n\to\infty} f(x_n) = f(x)$. □

Warning: it is *not* true that for a continuous function, $f(x_n) \to f(x) \Rightarrow x_n \to x$. Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ for all $x \in \mathbb{R}$, and $x_n = (-1)^n$. $f(x_n) \to f(x) = 1$ for *any* $x \in \mathbb{R}$, but that doesn't mean $x_n \to x$.

Exercise 7.4. $f : X \to Y$ is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. ◴

Proof. "⇒": $f\left(f^{-1}(V)\right) \subset V$.

"⇐": Given V open in Y, to prove $f^{-1}(V)$ is open in X, let $x \in f^{-1}(V)$. Then $f(x) \in V$, so that there is some neighborhood U of x such that $f(U) \subset V$. Then $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V)$. □

Exercise 7.5. $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed in X for every closed set B in Y. ◴

Proof. $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. □

Exercise 7.6. $f : X \to Y$ is continuous if and only if for every subset A of X, we have

$$f(\bar{A}) \subset \overline{f(A)}.$$ ◴

Proof. "⇒": $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$. By continuity, $f^{-1}(\overline{f(A)})$ is closed in X, so that $\bar{A} \subset f^{-1}(\overline{f(A)})$. Then $f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

"⇐": Let B be closed in Y, we prove $f^{-1}(B)$ is closed in X. By assumption, for the subset $f^{-1}(B)$ of X, we have $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B} = B$, so that $\overline{f^{-1}(B)} \subset f^{-1}(f(\overline{f^{-1}(B)})) \subset f^{-1}(B)$. Therefore, $f^{-1}(B)$ is closed. □

Exercise 7.7. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then there composite $g \circ f : X \to Z$ is continuous. ◴

Proof. Given U open in Z, $g^{-1}(U)$ is open in Y, and $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. □

7.1 Subbasis

Sometimes, we wish to form a topology on a set X from something even smaller than a basis.

> DEFINITION 7.8 (SUBBASIS) Let X be a set. Let $\mathcal{S} = \{S_\lambda\}$ be a collection of subsets of X such that
> $$X = \bigcup S_\lambda.$$
> Using $\mathcal{S}$, we can generate a topology on X as follows. First, collect all finite intersections of elements of $\mathcal{S}$, and note that the collection forms a basis. A topology $\mathcal{T}$ can be then generated from this basis. We call $\mathcal{S}$ a *subbasis* of $\mathcal{T}$, and we write $\mathcal{T} = \sigma(\mathcal{S})$.

For every $x \in U \in \mathcal{T}$, we have $x \in \left(S_1 \cap \cdots \cap S_n\right) \subset U$ for some $S_1, \dots, S_n \in \mathcal{S}$.

Exercise 7.9. Let $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$, where $\mathcal{T}_Y = \sigma(\mathcal{S})$ is generated by subbasis $\mathcal{S}$. Then f is continuous if and only if $f^{-1}(S) \in \mathcal{T}_X$ for every $S \in \mathcal{S}$. ◴

Proof. We prove f is continuous. Let $V \in \mathcal{T}_Y$. By Proposition 5.3, $V = \bigcup B_\lambda$ for some basis elements $\{B_\lambda\}$. Then
$$f^{-1}(V) = f^{-1}\left(\bigcup B_\lambda\right) = \bigcup f^{-1}(B_\lambda),$$
so that $f^{-1}(V)$ is open if every $f^{-1}(B_\lambda)$ is open.

Now, for each B_λ, $B_\lambda = S_1 \cap \cdots \cap S_n$ for some $S_1, \dots, S_n \in \mathcal{S}$. Then
$$f^{-1}(B_\lambda) = f^{-1}\left(\bigcap_{i=1}^{n} S_i\right) = \bigcap_{i=1}^{n} f^{-1}(S_i),$$
which is open since each $f^{-1}(S_i)$ is assumed to be open. □

> DEFINITION 7.10 (WEAK TOPOLOGY) Let $\{(Y, \mathcal{T}_\lambda)\}_{\lambda \in \Lambda}$ be a family of topological spaces, with functions $f_\lambda : X \to Y_\lambda$. Let
> $$\mathcal{S}_\lambda = \{f_\lambda^{-1}(V) \mid V \in \mathcal{T}_\lambda\}.$$
> Then $\mathcal{S} = \{\mathcal{S}_\lambda\}_{\lambda \in \Lambda}$ is a subbasis for X. The topology generated by this subbasis, $\mathcal{T} = \sigma(\mathcal{S})$, is called the *weak topology* on X with respect to $\{f_\lambda\}_{\lambda \in \Lambda}$. It is the coarsest topology on X such that each f_λ is continuous.

- Let X be a set, $(Y, \mathcal{T}_Y)$ be some topological space, and let $f : X \to Y$ be a constant function, i.e., $f(x) = c \in Y$ for all $x \in X$. Then for any $U \in \mathcal{T}_Y$,
$$f^{-1}(U) = \begin{cases} \varnothing & \text{if } c \notin U; \\ X & \text{if } c \in U. \end{cases}$$
Thus any constant function on X generates the trivial topology $\{\varnothing, X\}$ on X.

- Let $f : X \to Y$ has only two distinct values $\{c_1, c_2\}$ in Y. Then for any $U \in \mathcal{T}_Y$,

$$f^{-1}(U) = \begin{cases} f^{-1}(c_1) & \text{if } c_1 \in U, c_2 \notin U; \\ f^{-1}(c_2) & \text{if } c_2 \in U, c_1 \notin U; \\ \varnothing & \text{if } c_1, c_2 \notin U; \\ X & \text{if } c_1, c_2 \in U. \end{cases}$$

  Note that the two sets $f^{-1}(c_1)$ and $f^{-1}(c_2)$ are disjoint in X. Let $A = f^{-1}(c_1)$, then $f^{-1}(c_2) = X \setminus A$. The weak topology generated by the function f is then $\{\varnothing, X, A, X \setminus A\}$. It is the coarsest topology with respect to which f is continuous. The function f divides X into two disjoint "components".

- Similarly, for $f : X \to Y$ that takes three distinct values $\{c_1, c_2, c_3\}$ in Y, the coarsest topology with respect to which f is continuous is $\{\varnothing, X, A_1, X \setminus A_1, A_2, X \setminus A_2, A_3, X \setminus A_3\}$, where $A_i = f^{-1}(c_i)$ for $i = 1, 2, 3$. Similar construction can be made for any function that has n distinct values on X. Note that start from $n = 4$, it is not enough to only include A_i and $X \setminus A_i$ into the topology; for example, for $n = 4$, $A_1 \cup A_2 = (X \setminus A_3) \cap (X \setminus A_4) = X \setminus (A_3 \cup A_4)$.

- Equivalently, what we are doing above can also be seen as partitioning X into disjoint sets $A_1, A_2, \dots, A_n$, and we seek the coarsest topology on X relative to which each A_i is open.

DEFINITION 7.11 (SUBSPACE TOPOLOGY) Let $(X, \mathcal{T})$ be a topological space, and let $S \subset X$.

$$\iota : S \to X$$

is the inclusion map defined by $\iota(x) = x \in X$ for $x \in S$. The *subspace topology* $\mathcal{T}_S$ on S is defined to be the weak topology on S with respect to ι.

From the fact that $\iota^{-1}(U) = S \cap U$, Eq. (15) and Eq. (16), we see

$$\mathcal{T}_S = \{S \cap U : U \in \mathcal{T}\}.$$

The subspace topology is the coarsest topology on S for which ι is continuous. If we endow S with some different topologies, then the seemingly trivial map ι may fail to be continuous. For example, if we endow S with the trivial topology $\{\varnothing, S\}$, then ι may not be continuous.

Observation 7.12. Note that intersection communicates with Cartesian product, namely,

$$\bigcap_\beta \left(\prod_\alpha V_\alpha^\beta \right) = \prod_\alpha \left(\bigcap_\beta V_\alpha^\beta \right).$$

Thus,for example,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Let $\{X_i\}_{i=1}^\infty$ be a sequence of sets. If $U_1 \subset X_1$, $U_2 \subset X_2$, then

$$\left(U_1 \times X_2 \times X_3 \times \cdots\right) \cap \left(X_1 \times U_2 \times X_3 \times \cdots\right) = U_1 \times U_2 \times X_3 \times \cdots. \qquad \odot$$

Definition 7.13 (Product Topology) Let $\{(X_\lambda\ \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces. The *projection with index* λ_0, p_{λ_0}, is the function

$$p_{\lambda_0} : \prod_{\lambda\in\Lambda} X_\lambda \to X_{\lambda_0}$$

that maps an element of $\prod X_\lambda$ to its λ_0 component in X_{λ_0}. The *product topology* on $\prod X_\lambda$ is defined to be the weak topology on $\prod X_\lambda$ with respect to $\{p_\lambda\}_{\lambda\in\Lambda}$. It is the coarsest topology in which each projection is continuous.

We illustrate the definition using a sequence of topological spaces $\{(X_i, \mathcal{T}_i)\}_{i=1}^{\infty}$. Let $\mathcal{T}$ be the product topology on $\prod_{i=1}^{\infty} X_i$. To make each p_i continuous, we first put $\{p_i^{-1}(U), U \in \mathcal{T}_i\}$ for all i into $\mathcal{T}$. For example, for $U_1 \in \mathcal{T}_1$, $p_1^{-1}(U_1) = U_1 \times X_2 \times X_3 \times \cdots$; for $U_2 \in \mathcal{T}_2$, $p_2^{-1}(U_2) = X_1 \times U_2 \times X_3 \times \cdots$. Then we form the finite intersection of them to obtain our basis for $\mathcal{T}$. For example, according to our observation,

$$p_1^{-1}(U_1) \cap p_2^{-1}(U_2) \cap \cdots \cap p_n^{-1}(U_n) = U_1 \times U_2 \times \cdots \times U_n \times X_{n+1} \times X_{n+2} \times \cdots .$$

Thus, for every $U \in \mathcal{T}$, and every $x \in U$, there is a basis element $\prod_{i=1}^{\infty} U_i$ such that

$$x \in \prod_{i=1}^{\infty} U_i \subset U,$$

where $U_i \neq X_i$ for all but finitely many i.

Exercise 7.14. Let $A_\lambda \subset X_\lambda$ for each $\lambda \in \Lambda$. Then

$$\prod \bar{A}_\lambda = \overline{\prod A_\lambda}.$$ ◴

Proof. Since

$$\left(\prod U_\lambda\right) \cap \left(\prod A_\lambda\right) = \prod (U_\lambda \cap A_\lambda),$$

$\left(\prod U_\lambda\right) \cap \left(\prod A_\lambda\right) \neq \emptyset$ if and only if $U_\lambda \cap A_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$. □

Exercise 7.15. If each X_λ is a Hausdorff space, then $\prod_{\lambda\in\Lambda} X_\lambda$ is Hausdorff. ◴

Proof. Let $f \neq g \in \prod_{\lambda\in\Lambda} X_\lambda$. Then $f(\lambda_0) \neq g(\lambda_0)$ for some λ_0. Since X_{λ_0} is Hausdorff, there is some open sets U, V in X_{λ_0} such that $f(\lambda_0) \in U, g(\lambda_0) \in V$, and $U \cap V = \emptyset$. Then $f \in \prod U_\lambda$, where $U_\lambda = X_\lambda$ if $\lambda \neq \lambda_0$; $g \in \prod V_\lambda$, where $V_\lambda = X_\lambda$ if $\lambda \neq \lambda_0$, and

$$\left(\prod U_\lambda\right) \cap \left(\prod V_\lambda\right) = \emptyset.$$

To see this, if $h \in \left(\prod U_\lambda\right) \cap \left(\prod V_\lambda\right) = \prod \left(U_\lambda \cap V_\lambda\right)$, then $h(\lambda_0) \in \emptyset$, which is absurd. □

Theorem 7.16. *Let (f_n) be a sequence in $\prod_{\lambda\in\Lambda} X_\lambda$, where $f_n = \left(f_n(\lambda)\right)_{\lambda\in\Lambda}$, and let $f = (f(\lambda))_{\lambda\in\Lambda}$ be a point in this product space. Then $f_n \to f$ if and only if $f_n(\lambda) \to f(\lambda)$ in X_λ for each λ.* ◇

Proof. First let $f_n \to f$ in $\prod X_\lambda$. Then every neighborhood of f contains all but finitely many f_n. Given any λ_0, and any neighborhood U_{λ_0} of $f(\lambda_0)$, $\prod U_\lambda$, where $U_\lambda = X_\lambda$ for $\lambda \neq \lambda_0$, is a neighborhood of f, so that $f_n \in \prod U_\lambda$ for all but finitely many n. Then $f_n(\lambda_0) \in U_{\lambda_0}$ for all but finitely many n. This proves $f_n(\lambda_0) \to f(\lambda_0)$.

Now suppose $f_n(\lambda) \to f(\lambda)$ in X_λ for each λ. Does f_n converge to f? For each neighborhood U of f, there is a basis element B such that $f \in B \subset U$. Without loss of generality, and for the purpose of demonstration, we assume $B = U_{\lambda_1} \times \cdots \times U_{\lambda_N} \times \prod_{\lambda \neq \lambda_1, \ldots, \lambda_N} X_\lambda$. Then $f(\lambda_1) \in U_{\lambda_1}$, $f(\lambda_2) \in U_{\lambda_2}, \ldots \ldots$, and $f(\lambda_N) \in U_{\lambda_N}$. Since $f_n(\lambda) \to f(\lambda)$ in X_λ for each λ, we have $f_n(\lambda_1) \in U_{\lambda_1}$ for all but finitely many $n, \ldots \ldots, f_n(\lambda_N) \in U_{\lambda_N}$ for all but finitely many n. Then $f_n = \left(f_n(\lambda)\right)_{\lambda \in \Lambda} \in B$ for all but finitely many n. This proves $f_n \to f$. □

We mention that there is a second proof for the "only if " part. Since each projection p_λ is continuous, we have by Exercise 7.3

$$\lim_{n\to\infty} f_n(\lambda) = \lim_{n\to\infty} p_\lambda(f_n) = p_\lambda\left(\lim_{n\to\infty} f_n\right) = p_\lambda(f) = f(\lambda) \tag{17}$$

for each $\lambda \in \Lambda$.

- Let $\mathbb{R}^{[a,b]}$ denote the set of all real-valued functions defined on the interval $[a, b]$. Endow $\mathbb{R}^{[a,b]}$ with the product topology. Then (f_n) converges pointwise to f if and only if $f_n \to f$ in $\mathbb{R}^{[a,b]}$.

Theorem 7.17. *Let $f : (Y, \mathcal{T}) \to \prod_{\lambda\in\Lambda} X_\lambda$ be given by $f(y) = \left(f_\lambda(y)\right)_{\lambda\in\Lambda}$, where $f_\lambda : Y \to X_\lambda$ is the λ's component of f. Then f is continuous if and only if each f_λ is continuous.* ◇

Proof. Note that

$$f_\lambda = p_\lambda \circ f,$$

so that if f is continuous, then each f_λ is continuous, by Exercise 7.7. Conversely, suppose each f_λ is continuous. To prove f is continuous, we only need to verify that $f^{-1}\left(p_\lambda^{-1}(U)\right) \in \mathcal{T}$ for all λ and all U open in X_λ, by Exercise 7.9. But

$$f^{-1}\left(p_\lambda^{-1}(U)\right) = (p_\lambda \circ f)^{-1}(U) = f_\lambda^{-1}(U) \in \mathcal{T}.$$ □

Next we introduce the box topology, which is in some sense "dual" to the product topology.

DEFINITION 7.18 (BOX TOPOLOGY) Let $\{(X_\lambda \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$ be a family of topological spaces.

$$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} U_\lambda \mid U_\lambda \in \mathcal{T}_\lambda \right\}$$

is a basis on $\prod X_\lambda$. $\mathcal{T} = \sigma(\mathcal{B})$ is called the *box topology* on $\prod X_\lambda$.

Theorem 7.19. *Let $\{(X_\lambda \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$ be a family of T_1 spaces, and let $\mathcal{T}$ be the box topology on $\prod X_\lambda$. Suppose (f_n) is a sequence in $\prod X_\lambda$ and $f \in \prod X_\lambda$. Then $f_n \to f$ in $(\prod X_\lambda, \mathcal{T})$ if and only if*

1. *$f_n(\lambda) \to f(\lambda)$ in X_λ for each λ;*

2. *$\exists\Lambda_0$ finite, $f_n = f$ on $\Lambda \setminus \Lambda_0$ eventually.* ◇

Proof. "$\Leftarrow$": Let U be a neighborhood of f. Then there exists a basis element $\prod U_\lambda$ such that $f \in \prod U_\lambda \subset U$, where each U_λ is an open set in X_λ. For all those $\lambda \in \Lambda \setminus \Lambda_0$, there is some $N \in \mathbb{N}$ such that $f_n(\lambda) = f(\lambda) \in U_\lambda$ for all $n \geq N$. Now suppose Λ_0 has k elements. Then since $f_n(\lambda) \to f(\lambda)$ for each $\lambda \in \Lambda_0$,

$$\lambda_1 \in \Lambda_0 \Rightarrow \exists N_1 \in \mathbb{N},\ f_n(\lambda_1) \in U_{\lambda_1} \quad \forall n \geq N_1;$$
$$\lambda_2 \in \Lambda_0 \Rightarrow \exists N_2 \in \mathbb{N},\ f_n(\lambda_2) \in U_{\lambda_2} \quad \forall n \geq N_2;$$
$$\vdots$$
$$\lambda_k \in \Lambda_0 \Rightarrow \exists N_k \in \mathbb{N},\ f_n(\lambda_k) \in U_{\lambda_k} \quad \forall n \geq N_k.$$

Let $N' = \max\{N, N_1, \dots, N_k\}$. Then $f_n \in \prod U_\lambda$ for all $n \geq N'$. This proves $f_n \to f$.

"$\Rightarrow$": Box topology is finer than the product topology, so that in particular, each projection p_λ is continuous. Thus $f_n \to f$ implies $f_n(\lambda) = p_\lambda(f_n) \to p_\lambda(f) = f(\lambda)$, as in Eq. (17).

Now, is it necessary that $\exists\Lambda_0$ finite, $f_n = f$ on $\Lambda \setminus \Lambda_0$ eventually? Suppose, to the contrary, that

$$\forall \Lambda_0 \text{ finite, } f_n \neq f \text{ on } \Lambda \setminus \Lambda_0 \text{ infinitely often.}$$

Pick $n_1 \in \mathbb{N}$ and $\lambda_1 \in \Lambda$ such that $f_{n_1}(\lambda_1) \neq f(\lambda_1)$. Then there is a neighborhood V_{λ_1} of $f(\lambda_1)$ in X_{λ_1} such that $f_{n_1}(\lambda_1) \notin V_{\lambda_1}$. Similarly, pick $n_2 \in \mathbb{N}$ and $\lambda_2 \in \Lambda$ such that $f_{n_2}(\lambda_2) \neq f(\lambda_2)$. Then there is a neighborhood V_{λ_2} of $f(\lambda_2)$ in X_{λ_2} such that $f_{n_2}(\lambda_2) \notin V_{\lambda_2}$. Continuing this way, we obtain a sequence of open sets $\{V_{\lambda_1}, V_{\lambda_2}, \dots\}$. Let $V = \prod V_\lambda$, where we let V_λ be an arbitrary neighborhood of $f(\lambda)$ if $\lambda \notin \{\lambda_1, \lambda_2, \dots\}$. Then $f_n \notin V$ for infinitely many $n \in \mathbb{N}$, contrary to the assumption that $f_n \to f$. □

8 Connectedness

DEFINITION 8.1 Let $(X, \mathcal{T})$ be a topological space. It is called *disconnected* if there is a subset $A \notin \{\emptyset, X\}$ such that both $A, X \setminus A \in \mathcal{T}$. It is called *connected* if it is not disconnected.

Thus if X is disconnected, it can be written as a disjoint union of two open subsets.

Let

$$\mathcal{C} = \{A \subset X \mid X \setminus A \in \mathcal{T}\}$$

be the collection of all closed subsets in X. Since $X \setminus A \in \mathcal{T}$ if and only if $A \in \mathcal{C}$, and $A \in \mathcal{T}$ if and only if $X \setminus A \in \mathcal{C}$, we have both $A, X \setminus A \in \mathcal{T}$ if and only if both $A, X \setminus A \in \mathcal{C}$.

Corollary 8.2. *Let $\mathcal{C}$ denote the collection of closed subsets of $(X, \mathcal{T})$. X is disconnected if and only if $\mathcal{C} \cap \mathcal{T} \neq \{\emptyset, X\}$.* ♠

Table 1: Duality of Box Topology and Product Topology

Basis for product topology:	$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} U_\lambda \mid U_\lambda \in \mathcal{T}_\lambda \ \forall\lambda\in\Lambda;\ \exists\Lambda_0 \text{ finite}, U_\lambda = X_\lambda \text{ on } \Lambda\setminus\Lambda_0 \right\}$
Convergence in box topology:	$f_n \to f$ iff $f_n(\lambda) \to f(\lambda)\ \forall\lambda\in\Lambda;\ \exists\Lambda_0$ finite, $f_n = f$ on $\Lambda\setminus\Lambda_0$
Basis for box topology:	$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} U_\lambda : \quad U_\lambda \in \mathcal{T}_\lambda\ \ \forall\lambda\in\Lambda \right\}$
Convergence in product topology:	$f_n \to f$ iff $\quad f_n(\lambda) \to f(\lambda)\ \ \forall\lambda\in\Lambda$

Corollary 8.3. *X is disconnected if and only if there exists subset $A \notin \{\emptyset, X\}$ such that*

$$\bar{A} \cap (X \setminus A) = \emptyset \textit{ and } A \cap \overline{X \setminus A} = \emptyset. \qquad \spadesuit$$

Proof. If X is disconnected, then both $A, X\setminus A \in \mathcal{C}\cap\mathcal{T}$ so that the desired condition holds since both sets are closed. Conversely, from $\bar{A}\cap(X\setminus A) = \emptyset$ we have $\bar{A} \subset A$ so that $A \in \mathcal{C}$, and from $A \cap \overline{X\setminus A} = \emptyset$ we have $\overline{X\setminus A} \subset X\setminus A$, so that $X\setminus A \in \mathcal{C}$ as well. □

Theorem 8.4. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function that is surjective. If Y is disconnected, then X is disconnected.* ◇

Proof. By continuity, if both $A, Y\setminus A \in \mathcal{T}_Y$, then both $f^{-1}(A)$ and $f^{-1}(Y\setminus A) = X\setminus f^{-1}(A) \in \mathcal{T}_X$. □

Corollary 8.5. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function. If X is connected, then $f(X)$ is connected in Y.*

Proof. This is just the contrapositive of Theorem 8.4. □

Corollary 8.6. *$(X, \mathcal{T})$ is connected if and only if every continuous function $f : X \to \{0,1\}$ is constant, where $\{0,1\}$ has the discrete topology.* ♠

Proof. $\{0,1\}$ with the discrete topology is not connected, so that if X is connected, f cannot be surjective. Conversely, if both A and $X\setminus A \in \mathcal{T}$ for some nontrivial $A \subsetneq X$, then the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

is continuous and not constant. □

Corollary 8.7. *Let $\{A_\lambda\}$ be a collection of connected subspaces of X and assume $\bigcap A_\lambda \neq \varnothing$. Then $\bigcup A_\lambda$ is connected.* ♠

Proof. Let $p \in \bigcap A_\lambda$, and let $f : \bigcup A_\lambda \to \{0, 1\}$ be continuous. Then since each $\iota_\lambda \colon A_\lambda \to X$ is continuous, each $f \circ \iota_\lambda : A_\lambda \to \{0, 1\}$ is also continuous, thus constant, so that $f \circ \iota_\lambda(x) = f \circ \iota_\lambda(p) =$ constant for all λ and all $x \in \bigcup A_\lambda$. Thus f is constant on $\bigcup A_\lambda$, which proves $\bigcup A_\lambda$ is connected. □

Corollary 8.8. *If A is connected in X, then $\bar{A}$ is also connected in X.* ♠

Proof. Let $f : \bar{A} \to \{0, 1\}$ be continuous. Then by Exercise 7.6, $f(\bar{A}) \subset \overline{f(A)}$. Since $f \circ \iota_A : A \to \{0, 1\}$ is continuous, $f(A)$ is a singleton. Then $\overline{f(A)}$ is also the same singleton, which implies that $f(\bar{A})$ is a singleton. This proves every continuous $f : \bar{A} \to \{0, 1\}$ is constant. Hence $\bar{A}$ is connected. □

Proposition 8.9. *If X and Y are connected, then $X \times Y$ is connected.* ◆

Proof. Fix a point (a, b) in $X \times Y$. Then

$$X \times Y = \bigcup_{x \in X} (\{x\} \times Y) \cup (X \times \{b\}).$$

Each $(\{x\} \times Y) \cup (X \times \{b\})$ is connected, since the two connected spaces have (x, b) in common. Also, all of them contain (a, b), so that their union is connected by Corollary 8.7. □

From Proposition 8.9, a finite product of connected spaces is connected, by induction.

Proposition 8.10. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of connected spaces. Then $X = \prod X_\lambda$ is connected in the product topology.* ◆

Proof. Fix a point f in $\prod X_\lambda$. For any $\Lambda_0 \subset \Lambda$ finite, let

$$X_{\Lambda_0} = \{g \in \prod X_\lambda \mid g(\lambda) = f(\lambda) \text{ for } \lambda \notin \Lambda_0\}.$$

Each X_{Λ_0} is homeomorphic to a Cartesian product of finite connected spaces, hence connected. Since they all have the point f in common, there union

$$\bigcup_{\substack{\Lambda_0 \subset \Lambda \\ \text{finite}}} X_{\Lambda_0}$$

is connected. Now I claim

$$\prod X_\lambda = \overline{\bigcup_{\substack{\Lambda_0 \subset \Lambda \\ \text{finite}}} X_{\Lambda_0}},$$

so that connectedness of $\prod X_\lambda$ follows from Corollary 8.8. Let $h \in \prod X_\lambda$ be an arbitrary point, and let U be a neighborhood of h. Then there is a basis element $\prod U_\lambda$ such that $h \in \prod U_\lambda \subset U$. Further, there exists some finite Λ_0 in Λ such that $U_\lambda = X_\lambda$ for all $\lambda \in \Lambda \setminus \Lambda_0$. Then the point h', where

$$h'(\lambda) = \begin{cases} h(\lambda) & \lambda \in \Lambda_0 \\ f(\lambda) & \lambda \in \Lambda \setminus \Lambda_0 \end{cases}$$

is in X_{Λ_0}. This proves every neighborhood of h has a nonempty intersection with $\bigcup_{\Lambda_0 \subset \Lambda \text{ finite}} X_{\Lambda_0}$, so that its closure is indeed $\prod X_\lambda$ by Proposition 6.6. □

Theorem 8.11. *A real interval $[a, b]$ is connected. The real line $\mathbb{R}$ is thus connected. By Proposition 8.10, $\mathbb{R}^n$ is connected for any $n \in \mathbb{N}$.* ◇

Proof. Suppose $[a, b] = A \cup B$, where A and B are open and disjoint. Let $x = \sup A$. Then is $x \in A$ or $x \in B$? If $x \in A$, then there is some basis element (u, v) such that $x \in (u, v) \subset A$. Then since $x < v$, x is not an upper bound of A, a contradiction. If $x \in B$, then there is some basis element (u', v') such that $x \in (u', v') \subset B$. Note since $B \cap A = \emptyset$, we have $(u', v') \cap A = \emptyset$. Then u' would be an upper bound of A smaller than x, contradict to x being the least upper bound. □

On the other hand, $\mathbb{Q}$ is not connected. The set $U = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ is open in $\mathbb{Q}$, while its complement $\mathbb{Q} \setminus U = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$ is also open in $\mathbb{Q}$.

Theorem 8.12 (Intermediate Value Theorem). *Let $(X, \mathcal{T})$ be a connected space, and let $f : X \to \mathbb{R}$ be continuous. Suppose $f(x_1) \neq f(x_2)$ for some $x_1, x_2 \in X$, and with loss of generality suppose $f(x_1) < f(x_2)$. Then for every $r \in [f(x_1), f(x_2)] \subset \mathbb{R}$, there is some $x \in X$ such that $r = f(x)$.* ◇

Proof. Suppose no such x exists. Then for the open set $U = f^{-1}((-\infty, r))$ in X, its complement would be $X \setminus U = f^{-1}((r, +\infty))$, which is also open in X. This contradicts the fact that X is connected. □

Theorem 8.13 (Brouwer's Fixed-Point Theorem, One Dimension). *Every continuous $f : [0, 1] \to [0, 1]$ admits a fixed point, i.e., a point $x \in [0, 1]$ such that $f(x) = x$.* ◇

Proof. Since the range of f is $[0, 1]$, we have $f(0) \geq 0$ and $f(1) \leq 1$. If $f(0) = 0$ or $f(1) = 1$, we are done. Otherwise, $f(0) - 0 > 0$ and $f(1) - 1 < 0$, so that applying the Intermediate Value Theorem to the continuous function $f(x) - x$, we conclude that there is $x \in [0, 1]$ such that $f(x) - x = 0$. □

8.1 Path Connectedness

DEFINITION 8.14 Let $(X, \mathcal{T})$ be a topological space. Given two points $x, y \in X$, A *path* from x to y is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. X is called *path-connected* if there is a path between every pair of points in X.

Proposition 8.15. *If $(X, \mathcal{T})$ is path-connected, then X is connected.* ◆

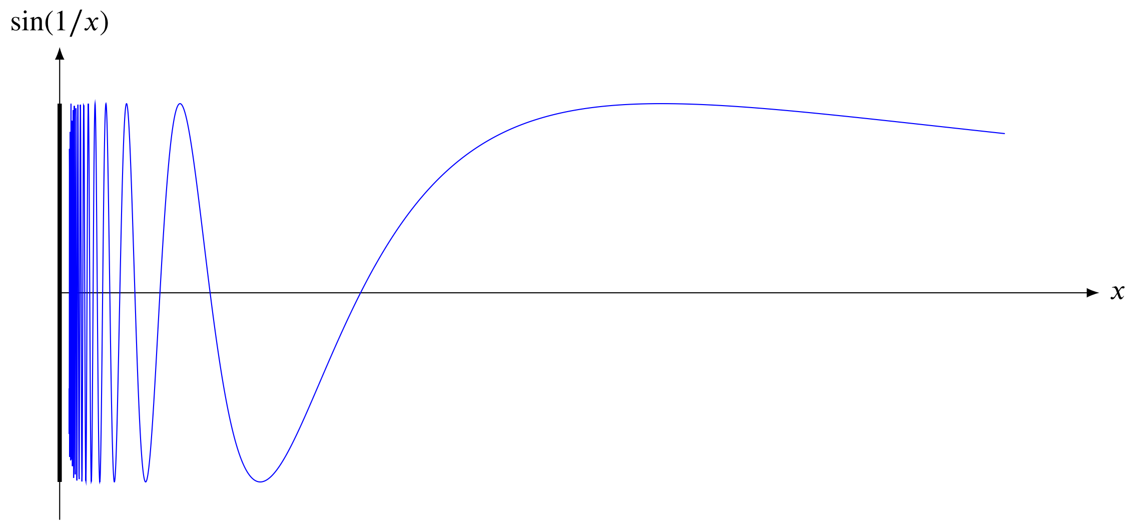


Figure 1: Topologist's sine curve

Proof. Suppose there is $A \in \mathcal{T}$ such that $X \setminus A \in \mathcal{T}$. Pick $x \in A$ and $y \in X \setminus A$. Let $\mathcal{P} = f([0,1])$ be the image in X of a path from x to y. Then $U = \mathcal{P} \cap A$ and $\mathcal{P} \setminus U = \mathcal{P} \cap (X \setminus A)$ would be both in $\mathcal{T}_{\mathcal{P}}$, the subspace topology for $\mathcal{P}$. But $\mathcal{P}$ is connected by Theorem 8.11 and Corollary 8.5. □

Example 8.16 (Topologist's Sine Curve). Consider $f : (0,1] \to \mathbb{R}^2$ given by $f(x) = (x, \sin(1/x))$. Its image is

$$S = \{(x, \sin(1/x)) \mid x \in (0,1]\}.$$

Since $(0,1]$ is connected, and f is continuous on $(0,1]$, S is connected in $\mathbb{R}^2$ by Corollary 8.5. Its closure

$$\bar{S} = \{(x, \sin(1/x)) \mid x \in (0,1]\} \cup (\{0\} \times [-1,1])$$

is thus also connected, by Corollary 8.8. $\bar{S}$ is called the *(closed) topologist's sine curve* (Fig. 1). ◁

Proposition 8.17. *$\bar{S}$ is not path-connected.* ◆

Proof. This is essentially the fact that the function

$$f(x) = \begin{cases} \sin(1/x) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

defined on $[0,1]$ is not continuous. This is because we can find a sequence (x_n) in $[0,1]$ such that $x_n \to 0$ but $f(x_n)$ does not converge. To construct a specific example, let

$$x_n = \frac{1}{\frac{(2n+1)\pi}{2}}$$

so that

$$f(x_n) = \sin\frac{(2n+1)\pi}{2} = \begin{cases} 1 & n \text{ even;} \\ -1 & n \text{ odd.} \end{cases}$$

Any other function $f : [0,1] \to \bar{S}$ such that $f(1) \in S$ would fail to be continuous for similar reasons, so that there does not exist a path from $(0,0)$ to a point in S. $\bar{S}$ is thus not path-connected. □

DEFINITION 8.18 (PATH COMPONENTS) Let $(X, \mathcal{T})$ be a topological space. Define an equivalent relation by setting $x \sim y$ if there is a path from x to y. The equivalent classes are called the *path components* of X.

- There are two path components of the topologist's sine curve $\bar{S}$. One is S, and the other is $\{0\} \times [-1,1]$. Note that S is open but not closed, while $\{0\} \times [-1,1]$ is closed but not open.

DEFINITION 8.19 (COMPONENTS) Let $(X, \mathcal{T})$ be a topological space. Define an equivalent relation by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalent classes are called the *(connected) components* of X.

Proposition 8.20. *Each component in X is closed.* ◆

Proof. Let $A \subset X$ be an arbitrary connected subspace. Then $x \sim y$ for any $x, y \in A$, so that A falls in a single component. Thus, components are maximal connected subspaces in X. Since the closure of a connected subspace is connected by Corollary 8.8, each component is closed. □

- If X has only finitely many components, then each of them is also open in X. For example, let $X = C_1 \cup C_2 \cup \cdots \cup C_n$. $C_2 \cup \cdots \cup C_n$ is closed, being a finite union of closed subsets. Hence $C_1 = X \setminus (C_2 \cup \cdots \cup C_n)$ is open. On the other hand, if X has infinitely many components, then they may not be open. For example, the components of $\mathbb{Q}$ are one-point sets $\{q\}$, and each such singleton is not open.

DEFINITION 8.21 (QUASICOMPONENTS) Let $(X, \mathcal{T})$ be a topological space. Define an equivalent relation by setting $x \sim y$ if there is no $x \in A \in \mathcal{T}$ such that $y \in X \setminus A \in \mathcal{T}$. The equivalent classes are called the *quasicomponents* of X.

Proposition 8.22. *Each quasicomponent in X is closed.* ◆

Proof. Let $x \in X$ and let $[Q_x]$ denote the quasicomponent that contains x. Then it is easy to see that

$$[Q_x] = \{y \in X \mid \text{no } x \in A \in \mathcal{C} \cap \mathcal{T} \text{ such that } y \notin A\}.$$

Thus, $y \in [Q_x]$ if and only if $y \in A$ for all $A \in \mathcal{C} \cap \mathcal{T}$. Thus

$$[Q_x] = \bigcap_{\substack{x \in A \\ A \in \mathcal{C} \cap \mathcal{T}}} A. \tag{18}$$

Each such A is closed, so that their intersection is closed. □

Observation 8.23. Let $x \in X$, and let $[\mathcal{P}_x]$, $[\mathcal{C}_x]$ and $[\mathcal{Q}_x]$ denote the path component, component, and quasicomponent that contain x, respectively. Then

$$[\mathcal{P}_x] \subset [\mathcal{C}_x] \subset [\mathcal{Q}_x]. \qquad \odot$$

8.2 Local Connectedness

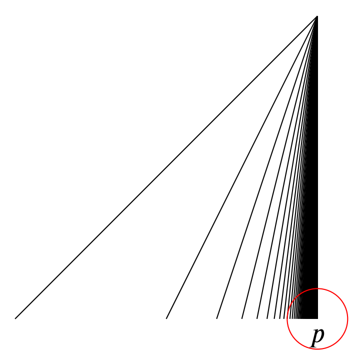
DEFINITION 8.24 $(X, \mathcal{T})$ is *locally connected at x* if every neighborhood of x contains a connected neighborhood. X is called *locally connected* if it is locally connected at each of its points.

8.2.1 Examples

- It is immediate that, if $\mathcal{T} = \sigma(\mathcal{B})$, where each $B \in \mathcal{B}$ is connected, then $(X, \mathcal{T})$ is locally connected.
- $A = [1, 2] \cup [3, 4]$ is locally connected, but not connected.
- The topologist's sine curve is connected, but not locally connected. No connected neighborhood is contained in any neighborhood of $(0, 0)$.
- $\mathbb{Q}$ is neither connected nor locally connected.

DEFINITION 8.25 X is *locally path connected at x* if every neighborhood of x contains a path connected neighborhood. X is called *locally path connected* if it is locally path connected at each of its points.

- $A = [1, 2] \cup [3, 4]$ is locally path connected, but not path connected.
- Consider this infinite broom. It is path connected, but not locally path connected. Every neighborhood of p would enclose infinitely many disjoint "branches".



Proposition 8.26. *$(X, \mathcal{T})$ is locally connected if and only if for every $U \in \mathcal{T}$, every component $[\mathcal{C}_x]$ in U is in $\mathcal{T}$. In particular, if X is locally connected, then each component of X is open.* ♦

Proof. It is clear that if for every $U \in \mathcal{T}$ and $[\mathcal{C}_x] \subset U$, $[\mathcal{C}_x] \in \mathcal{T}$ for every $x \in U$, then X is locally connected. Every $x \in U \in \mathcal{T}$ contains a connected neighborhood (which is $[\mathcal{C}_x]$).

Conversely, suppose $(X, \mathcal{T})$ is locally connected. Let $U \in \mathcal{T}$, and let $[\mathcal{C}_x] \subset U$ be a component of U. If $y \in [\mathcal{C}_x]$, then there exists $V \in \mathcal{T}$ such that $y \in V \subset U$ and V is connected. Since $V \cap [\mathcal{C}_x] \neq \varnothing$, we have $V \subset [\mathcal{C}_x]$. This proves $[\mathcal{C}_x]$ is open. □

Proposition 8.27. *$(X, \mathcal{T})$ is locally path connected if and only if for every $U \in \mathcal{T}$, every path component $[\mathcal{P}_x]$ in U is in $\mathcal{T}$. In particular, if X is locally path connected, then each path component of X is open.* ♦

Proof. Similar to the proof of Proposition 8.26. □

Proposition 8.28. *If X is locally connected, then $[\mathcal{C}_x] = [\mathcal{Q}_x]$ for every $x \in X$.* ♦

Proof. By Observation 8.23, $[\mathcal{C}_x] \subset [\mathcal{Q}_x]$. We prove $[\mathcal{Q}_x] \subset [\mathcal{C}_x]$. By Proposition 8.26, $[\mathcal{C}_x] \in \mathcal{T}$ for every $x \in X$. $[\mathcal{C}_x]$ is also in $\mathcal{C}$ by Proposition 8.20, so that $[\mathcal{C}_x] \in \mathcal{C} \cap \mathcal{T}$. Then referring to Eq. (18), we see that

$$[\mathcal{Q}_x] = \bigcap_{\substack{x \in A \\ A \in \mathcal{C} \cap \mathcal{T}}} A \subset [\mathcal{C}_x].$$

□

Proposition 8.29. *If X is locally path connected, then $[\mathcal{P}_x] = [\mathcal{C}_x]$ for every $x \in X$.* ♦

Proof. By Observation 8.23, $[\mathcal{P}_x] \subset [\mathcal{C}_x]$. Suppose $[\mathcal{P}_x] \subsetneq [\mathcal{C}_x]$. We let $\mathcal{P}$ be the collection of other path components in X that have nonempty intersection with $[\mathcal{C}_x]$, i.e.,

$$\mathcal{P} = \left\{ [\mathcal{P}_y] \mid [\mathcal{P}_y] \neq [\mathcal{P}_x],\ [\mathcal{P}_y] \cap [\mathcal{C}_x] \neq \varnothing \right\}.$$

Since each $[\mathcal{P}_y] \in \mathcal{P}$ is connected, $[\mathcal{P}_y] \subset [\mathcal{C}_x]$. Thus,

$$[\mathcal{C}_x] = [\mathcal{P}_x] \cup \left(\bigcup_{[\mathcal{P}_y] \in \mathcal{P}} [\mathcal{P}_y] \right),$$

where $\bigcup_{[\mathcal{P}_y] \in \mathcal{P}} [\mathcal{P}_y] = [\mathcal{C}_x] \setminus [\mathcal{P}_x]$ is open since each $[\mathcal{P}_y]$ is open by Proposition 8.27. We arrived at a contradiction that $[\mathcal{C}_x]$ is disconnected. □

Corollary 8.30. *If X is locally path connected, then $[\mathcal{P}_x] = [\mathcal{C}_x] = [\mathcal{Q}_x]$ for every $x \in X$.* ♠

9 Compactness

We generalize the notion of compactness in metric spaces to general topological spaces.

DEFINITION 9.1 Let $(X, \mathcal{T})$ be a topological space. A subset A of X is said to be *compact* if every open cover of A has a finite subcover. Namely, for any collection $\{U_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{T}$ such that $A \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, there exists $\Lambda_0 \subset \Lambda$ finite such that

$$A \subset \bigcup_{\lambda\in\Lambda_0} U_\lambda \subset \bigcup_{\lambda\in\Lambda} U_\lambda.$$

Observation 9.2. If $\mathcal{T} = \sigma(\mathcal{B})$, then to check X is compact, we may only need to check every open cover by elements in $\mathcal{B}$ has a finite subcover. ⊙

Theorem 9.3. *The real interval* $[a, b]$ *is compact in* $(\mathbb{R}, \mathcal{T})$*, where* $\mathcal{T}$ *is the usual topology.* ◇

Proof. Let $\{U_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{T}$ such that $[a, b] \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, and let

$$A = \{x \in [a, b] \mid [a, x] \subset \bigcup_{\lambda\in\Lambda_0} U_\lambda \text{ for some } \Lambda_0 \subset \Lambda \text{ finite } \}.$$

Let $x_0 = \sup A$. Is $x_0 = b$? Suppose $x_0 < b$.

Then since $x_0 \in U_{\lambda'}$ for some $U_{\lambda'} \in \mathcal{T}$, there is a basis element $(x_0 - \epsilon, x_0 + \epsilon)$ such that $x_0 \in (x_0 - \epsilon, x_0 + \epsilon) \subset U_{\lambda'}$. Since $x_0 - \epsilon$ is not an upper bound for A, there is some $z \in A$ such that $x_0 - \epsilon < z \leq x_0$, so that $[a, z] \subset \bigcup_{\lambda\in\Lambda_0} U_\lambda$ for some $\Lambda_0 \subset \Lambda$ finite. Also, pick some $y \in (x_0, x_0 + \epsilon)$. Then

$$[a, y] \subset \left(\bigcup_{\lambda\in\Lambda_0} U_\lambda\right) \cup U_{\lambda'},$$

so that $x_0 < y \in A$, contrary to the fact that x_0 is an upper bound of A. □

Proposition 9.4. *If* $(X, \mathcal{T})$ *is compact, then any* $A \in \mathcal{C}$ *is compact.* ◆

Proof. Let $\mathcal{O}$ be an open cover of A. Then since $X \setminus A \in \mathcal{T}$, $\mathcal{O} \cup (X \setminus A)$ is an open cover of X. By compactness of X, there is some finite $\mathcal{O}'$ that covers X. Then $\mathcal{O}' \setminus (X \setminus A)$ is a finite cover of A. □

Proposition 9.5. *If* $(X, \mathcal{T})$ *is Hausdorff, then any compact subspace of* X *is closed.* ◆

Recall we have proved in the lecture that any compact subset of a metric space is closed and bounded. Note how the proof here is similar to the proof for metric space.

Proof. Let A be compact; we prove $X \setminus A$ is open. Pick $x_0 \in X \setminus A$. Then for every $y \in A$, there are U_y and $V_y \in \mathcal{T}$ such that $x_0 \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. $\{V_y\}_{y\in A}$ is an open cover of A; by compactness of A, $A \subset \bigcup_{i=1}^n V_{y_i} =: V$ for some finite number of points $y_1, \dots, y_n$ in A. Then

$$U := \left(\bigcap_{i=1}^n U_{y_i}\right) \cap V = \emptyset.$$

This proves that for every $x \in X \setminus A$, we can find a neighborhood U of x such that $x \in U \subset X \setminus A$. Thus $X \setminus A$ is open, so that A is closed. □

Proposition 9.6. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous. If $A \subset X$ is compact, then $f(A)$ is compact in Y.* ◆

Recall the first question in Problem Set 2.

Proof. Let

$$f(A) \subset \bigcup_{\lambda \in \Lambda} V_\lambda,$$

where $\{V_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{T}_Y$. Then

$$A \subset f^{-1}(f(A)) \subset f^{-1}\left(\bigcup_{\lambda \in \Lambda} V_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda).$$

A compact $\Rightarrow A \subset \bigcup_{\lambda \in \Lambda_0} f^{-1}(V_\lambda) \subset \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda)$ for some $\Lambda_0 \subset \Lambda$ finite. Then

$$f(A) \subset f\left(\bigcup_{\lambda \in \Lambda_0} f^{-1}(V_\lambda)\right) = \bigcup_{\lambda \in \Lambda_0} f\left(f^{-1}(V_\lambda)\right) \subset \bigcup_{\lambda \in \Lambda_0} V_\lambda.$$

□

Corollary 9.7. *If X is compact, then any continuous function $f : X \to \mathbb{R}$ is bounded.* ♠

Proof. $f(X)$ is compact in $\mathbb{R}$, hence bounded. □

Theorem 9.8 (Extreme Value Theorem). *Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then $\inf f(X) \in f(X)$ and $\sup f(X) \in f(X)$.* ◇

Proof. $f(X)$ is compact in $\mathbb{R}$, hence closed. □

We next prove the Uniform Continuity Theorem, which says that a continuous function on a compact metric space is uniformly continuous. To prove the theorem, we will first need a lemma, the Lebesgue Number Lemma. Recall that the *diameter* of a set $E \subset X$ is defined to be

$$\text{diam}(E) = \sup\{d(x, y) \mid x, y \in E\}.$$

Lemma 9.9 (Lebesgue Number Lemma). *Let (X, d) be a compact metric space. Then given any open cover $\mathcal{O}$ of X, there is some $\delta > 0$ such that*

$$\forall E \subset X, \;\; \text{diam}(E) < \delta \;\Rightarrow\; E \subset U \textit{ for some } U \in \mathcal{O}.$$

The number δ is called the Lebesgue number *for $\mathcal{O}$.* ✻

Proof. For any $x \in X$, there is some $U(x)$ in $\mathcal{O}$ such that $x \in U(x)$. Then there is some open ball $B(x, \epsilon_x)$ around x such that $x \in B(x, \epsilon_x) \subset U(x)$. The collection of open balls

$$\{B(x, \epsilon_x/2)\}_{x \in X}$$

is thus an open cover of X, so by compactness

$$X \subset B(x_1, \epsilon_{x_1}/2) \cup \cdots \cup B(x_n, \epsilon_{x_n}/2)$$

for some finite set of points $x_1, \ldots, x_n$ in X. Let $\delta = \min\{\epsilon_{x_1}/2, \ldots, \epsilon_{x_n}/2\}$. We claim δ is the desired Lebesgue number. Let $E \subset X$ such that $\operatorname{diam}(E) < \delta$, and fix a point $p \in E$. Then $p \in B(x_i, \epsilon_{x_i}/2)$ for some x_i, so that $d(x_i, p) < \epsilon_{x_i}/2$. Let $x \in E$. Then

$$d(p, x) \leq \operatorname{diam}(E) < \delta \leq \epsilon_{x_i}/2,$$

so that

$$d(x_i, x) \leq d(x_i, p) + d(p, x) < \epsilon_{x_i}/2 + \epsilon_{x_i}/2 = \epsilon_{x_i}.$$

This shows

$$E \subset B(x_i, \epsilon_{x_i}/2) \subset U(x_i).$$

□

Theorem 9.10 (Uniform Continuity Theorem). *Let $f : (X, d_X) \to (Y, d_Y)$ be continuous. If X is compact, then f is uniformly continuous.* ◇

Proof. Given $\epsilon > 0$, $\{B(y, \epsilon/2)\}_{y \in Y}$ is an open cover of Y. Since f is continuous,

$$\{f^{-1}(B(y, \epsilon/2))\}_{y \in Y}$$

is an open cover of X, which admits a Lebesgue number $\delta > 0$ by Lemma 9.9. Then given any $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, $x_1, x_2 \in f^{-1}(B(y, \epsilon/2))$ for some $y \in Y$. Then $f(x_1), f(x_2) \in B(y, \epsilon/2)$, so that $d_Y(f(x_1), f(x_2)) < \epsilon$. □

Proposition 9.11. *If X and Y is compact, then $X \times Y$ is compact.* ◆

Proof. Let $\mathcal{O} = \{U_\lambda \times V_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X \times Y$ by basis elements. For each $(x, y) \in X \times Y$, there is some $U(x, y) \times V(x, y) \in \mathcal{O}$ such that $(x, y) \in U(x, y) \times V(x, y)$. Since $\{x\} \times Y$ is compact, we have

$$\{x\} \times Y \subset \bigcup_{i=1}^{n} U(x, y_i) \times V(x, y_i)$$

for some $y_1, \ldots, y_n$ in Y. Put

$$U_x = \bigcap_{i=1}^{n} U(x, y_i),$$

and note that $U_x \times Y \subset \bigcup_{i=1}^{n} U(x, y_i) \times V(x, y_i)$. Now $\{U_x \times Y\}_{x \in X}$ is an open cover of $X \times \{y\}$ for any $y \in Y$, so by compactness

$$X \times \{y\} \subset \bigcup_{j=1}^{m} U_{x_j} \times Y$$

for some $x_1, \ldots, x_m$ in X. Then

$$X \times Y \subset \bigcup_{j=1}^{m} U_{x_j} \times Y.$$

Since each $U_{x_j} \times Y$ can be covered by finitely many elements in $\mathcal{O}$, $X \times Y$ can be covered by finitely many elements in $\mathcal{O}$. □

Corollary 9.12. *A finite product of compact spaces is compact. In particular, $[a,b]^n$ is compact in $\mathbb{R}^n$.* ♠

Proof. Proposition 9.11. □

Corollary 9.13. *$A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.* ♠

Proof. We have proved in class that a compact subset of a metric space is closed and bounded. Conversely, if A is closed and bounded, then it is contained in $[-N,N]^n$ for some $N > 0$, which is compact by Corollary 9.12. Since a closed subset of a compact set is compact, A is compact. □

It turns out that an arbitrary product of compact spaces is also compact in the product topology. This is the Tychonoff's Theorem. It is equivalent to the axiom of choice. To prove the theorem, we first need a lemma, the Alexander Subbasis Theorem.

Theorem 9.14 (Alexander Subbasis Theorem). *Let $(X,\mathcal{T})$ be a topological space, where $\mathcal{T} = \sigma(\mathcal{S})$ is generated by a subbasis $\mathcal{S}$. Then X is compact if and only if every open cover from $\mathcal{S}$ has a finite subcover.* ◇

Proof. Suppose every open cover from $\mathcal{S}$ has a finite subcover, yet X is not compact. By Observation 9.2, there is some open cover by the the basis elements in $\mathcal{B}(\mathcal{S})$, generated by $\mathcal{S}$, that has no finite subcover. Let $\mathcal{J}$ be the collection of all such covers, and note that set inclusion is a partial order on $\mathcal{J}$, where an upper bound for each chain is the union of all the covers in the chain. By Zorn's lemma, $\mathcal{J}$ has a maximal element $\mathcal{O} = \{U_\lambda\}_{\lambda\in\Lambda}$. $\mathcal{O}$ has no finite subcover of X, but $\mathcal{O} \cup \{U'\}$ for any other $U' \in \mathcal{B}(\mathcal{S})$ would have a finite subcover.

Pick an arbitrary $U_\lambda \in \mathcal{O}$, so that $U_\lambda = S_1 \cap \cdots \cap S_n$ for some $S_1, \ldots, S_n \in \mathcal{S}$. Then $S_i \in \mathcal{O}$ for at least one $i \in \{1,\ldots,n\}$. For if not, $\mathcal{O} \cup \{S_1\}, \ldots, \mathcal{O} \cup \{S_n\}$ would all have a finite subcover of X, so that $\mathcal{O}$ has a finite subcover for $X \setminus S_i$ for each $i = 1.\ldots,n$. But then $\mathcal{O}$ would have a finite cover for

$$X \setminus U_\lambda = X \setminus \left(\bigcap_{i=1}^{n} S_i\right) = \bigcup_{i=1}^{n}(X \setminus S_i),$$

and thus for $X = (X \setminus U_\lambda) \cup U_\lambda$, a contradiction.

Thus, for any $U_\lambda \in \mathcal{O}$, there is $U_\lambda \subset S_\lambda \in \mathcal{O}$ for some $S_\lambda \in \mathcal{S}$. Then

$$X \subset \bigcup_{\lambda\in\Lambda} U_\lambda \subset \bigcup_{\lambda\in\Lambda} S_\lambda.$$

By assumption, the cover $\{S_\lambda\}_{\lambda\in\Lambda}$ has a finite subcover. But since $\{S_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{O}$, this implies $\mathcal{O} \notin \mathcal{J}$, a contradiction. □

Theorem 9.15 (Tychonoff's Theorem). *Let $\{(X_\lambda, \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$ be a family of compact spaces. Then $\prod_{\lambda\in\Lambda} X_\lambda$ is compact in the product topology.* ◇

Proof. Let

$$\mathcal{O} = \{p_\lambda^{-1}\left(U_\lambda^\alpha\right) : \alpha \in A(\lambda), \lambda \in \Lambda\}$$

be an arbitrary open cover of $\prod_{\lambda\in\Lambda} X_\lambda$ by subbasis elements. If for some $\lambda \in \Lambda$,

$$\{U_\lambda^\alpha : \alpha \in A(\lambda)\}$$

covers X_λ, then by compactness of X_λ,

$$X_\lambda \subset U_\lambda^{\alpha_1} \cup \cdots \cup U_\lambda^{\alpha_n}$$

for some finite $\alpha_1, \ldots, \alpha_n \in A(\lambda)$. In this case

$$\prod_{\lambda\in\Lambda} X_\lambda \subset p_\lambda^{-1}\left(U_\lambda^{\alpha_1}\right) \cup \cdots \cup p_\lambda^{-1}\left(U_\lambda^{\alpha_n}\right),$$

and we are done. If for all $\lambda \in \Lambda$, none of $\{U_\lambda^\alpha : \alpha \in A(\lambda)\}$ covers X_λ, then we can select

$$f(\lambda) \notin \bigcup_{\alpha\in A(\lambda)} U_\lambda^\alpha$$

for each $\lambda \in \Lambda$, obtaining an $f \in \prod_{\lambda\in\Lambda} X_\lambda$. But then $f \notin \mathcal{O}$, contrary to the assumption that $\mathcal{O}$ covers $\prod_{\lambda\in\Lambda} X_\lambda$. □

Our definition of compactness is in terms of open sets. An equivalent definition can be formulated using closed sets.

Proposition 9.16. *$(X, \mathcal{T})$ is compact if and only if for every $\{C_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{C}$,*

$$\bigcap_{\lambda\in\Lambda_0} C_\lambda \neq \varnothing \text{ for any finite } \Lambda_0 \subset \Lambda \;\Rightarrow\; \bigcap_{\lambda\in\Lambda} C_\lambda \neq \varnothing.$$

◆

Proof. $X \subset \bigcup_{\lambda\in\Lambda} U_\lambda \Rightarrow X \subset \bigcup_{\lambda\in\Lambda_0} U_\lambda$ for some $\Lambda_0 \subset \Lambda$ finite if and only if

$$\varnothing = X \setminus X \supset X \setminus \left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcap_{\lambda\in\Lambda} \left(X \setminus U_\lambda\right)$$

$$\Downarrow$$

$$\varnothing = \bigcap_{\lambda\in\Lambda_0} \left(X \setminus U_\lambda\right) \text{ for some } \Lambda_0 \subset \Lambda \text{ finite.}$$

Now take the contrapositive. □

DEFINITION 9.17 Several other notions of compactness:

1. A space X is said to be *σ-compact* if it is the union of countably many compact sets.
2. A space X is said to be *Lindelöf* if every open cover of X has a countable subcover.
3. A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.
4. A space X is said to be *countably compact* if every countable open cover of X has a finite subcover.
5. A space X is said to be *limit point compact* if every infinite subset of X has a limit point.
6. A space X is said to be *pseudocompact* if every continuous real-valued function on$X$ is bounded.

Observation 9.18. Compact ⇒ σ-compact ⇒ Lindelöf; compact ⇒ countably compact. ⊙

Exercise 9.19. X is countably compact if and only if every nested sequence of closed nonempty sets $C_1 \supset C_2 \supset \cdots$ has nonempty intersection. ◴

Proof. Proposition 9.16. □

Exercise 9.20. If X is sequentially compact, then it is countably compact. ◴

Proof (Proof 1). Let $C_1 \supset C_2 \supset \cdots$ be a nested closed nonempty sets in X, and select $x_n \in C_n$ for each n. By assumption, (x_n) has a subsequence (x_{n_k}) that converges to $x \in X$. Consider an C_n, which contains $\{x_i : i \geq n\}$. Then for each neighborhood U of x, $U \cap C_n$ contains infinitely many points, so that $x \in \bar{C}_n = C_n$. This is true for all $n \in \mathbb{N}$, so that $x \in C_n$ for all $n \in \mathbb{N}$. We thus have $x \in \bigcap_{n=1}^{\infty} C_n$.□

Proof (Proof 2). Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable open cover of X, and suppose it has no finite subcover. Then we can select $x_n \notin \bigcup_{i=1}^{n} U_n$ for each $n \in \mathbb{N}$, obtaining a sequence (x_n) in X. Since X is assumed to be sequentially compact, (x_n) has a subsequence (x_{n_k}) that converges to some $x \in X$. Since $\{U_n\}_{n\in\mathbb{N}}$ covers X, we have $x \in U_m \subset \bigcup_{i=1}^{m} U_m$ for some $m \in \mathbb{N}$. But $\bigcup_{i=1}^{m} U_m$ would contain infinitely many items of (x_n), contrary to our selection of the sequence (x_n). □

Exercise 9.21. Every countably compact space X is limit point compact; the converse holds if X is T_1. ◴

Proof. Suppose first that X is countably compact. Let $A \subset X$ be an infinite set. Suppose, to the contrary, that A has no limit point. Then any subset of A has no limit point, so that any subset of A is closed. Pick a sequence of distinct points (x_n) in A. Then $\{O_n\}_{n\in\mathbb{N}}$, where $O_n = X \setminus \{x_n, x_{n+1}, \dots\}$, is a countable open cover of X. But this open cover can not have a finite subcover, for if $X \subset \bigcup_{n=1}^{N} O_n = O_N$, then we would have $X \subset X \setminus \{x_N, x_{N+1}, \dots\}$, which is absurd.

Suppose now X is limit point compact and T_1. If there is some countable open cover $\{U_n\}_{n\in\mathbb{N}}$

of X that does not have a finite subcover, we can pick $x_n \in X \setminus (U_1 \cup \cdots \cup U_n)$ for each $n \in \mathbb{N}$. Then the infinite set $A = \{x_n : n \in \mathbb{N}\}$ would not have any limit point: any $x \in X$ lies in some U_N, so in particular intersects A at only finitely many points, so that it can't be a limit point of A by Exercise 6.17. □

Exercise 9.22. Every countably compact space X is pseudocompact. ◴

Proof. Let X be countably compact, and let $f : X \to \mathbb{R}$ be a continuous real-valued function. Then $X \subset \bigcup_{n=1}^{\infty} O_n$, where $O_n = \{x \in X : |f(x)| < n\}$. A finite subcover would mean $X \subset \bigcup_{n=1}^{N} O_n = O_N$, so that $|f(x)| < N$ for all $x \in X$. □

Theorem 9.23. *For a metric space (X, d), 3, 4, 5, 6 in Definition 9.17 are all equivalent to compactness.* ◇

Proof. By Exercise 9.20, Exercise 9.21, and Exercise 9.22, to prove the theorem, it remains to prove

1. limit point compact implies sequentially compact for (X, d);
2. pseudocompact implies sequentially compact for (X, d), and
3. sequentially compact implies compact for (X, d).

Proof (Limit point compact ⇒ sequentially compact). Suppose (X, d) is limit point compact, and let (x_n) be a sequence of distinct points in X. The infinite set $A = \{x_n : n \in \mathbb{N}\}$ will then have a limit point $x \in X$. Since (X, d) is Hausdorff and hence T_1, $B(x, \epsilon) \cap A$ contains infinitely many points of A for any $\epsilon > 0$. Then we can pick $x_{n_k} \in B(x, \frac{1}{k})$ for $k = 1, 2, \ldots$, thus obtain a subsequence (x_{n_k}) that converges to x. □

Proof (Pseudocompact ⇒ sequentially compact). Suppose (X, d) is pseudosompact, and let (x_n) be a sequence in X that does not have a convergent subsequence. Then $A = \{x_n : n \in \mathbb{N}\}$ is discrete, so for every x_n there exists ϵ_n such that $\bar{B}(x_n, \epsilon_n) \cap \bar{B}(x_m, \epsilon_m) = \emptyset$ for all $n \neq m$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} n\left(1 - \frac{d(x, x_n)}{\epsilon_n}\right) & x \in B(x_n, \epsilon_n); \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f is continuous but not bounded. □

Proof (Sequentially compact ⇒ compact). Suppose (X, d) is sequentially compact. Then

(a) (X, d) satisfies the Lebesgue number lemma.

Suppose not; then for some open cover $\mathcal{O}$ of X, and for every $n \in \mathbb{N}$, there is $E_n \subset X$ with $\operatorname{diam}(E_n) < \frac{1}{n}$ such that $E_n \not\subseteq U$ for any $U \in \mathcal{O}$. Pick $x_n \in E_n$ for each $n \in \mathbb{N}$. By assumption, the sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \to x_0$ for some $x_0 \in X$. $x_0 \in U$ for some $U \in \mathcal{O}$, so there is some $\epsilon > 0$ such that $x_0 \in B(x_0, \epsilon) \subset U$. Since $\operatorname{diam}(E_n) \to 0$, there is $N_1 \in \mathbb{N}$ such that $\operatorname{diam}(E_{n_k}) < \frac{\epsilon}{2}$ for $n_k \geq N_1$; and since $x_{n_k} \to x_0$, there is $N_2 \in \mathbb{N}$ such that $d(x_{n_k}, x_0) < \frac{\epsilon}{2}$ for $n_k \geq N_2$. Let $N = \max\{N_1, N_2\}$, $n_k \geq N$, and let $x \in E_{n_k}$. Then

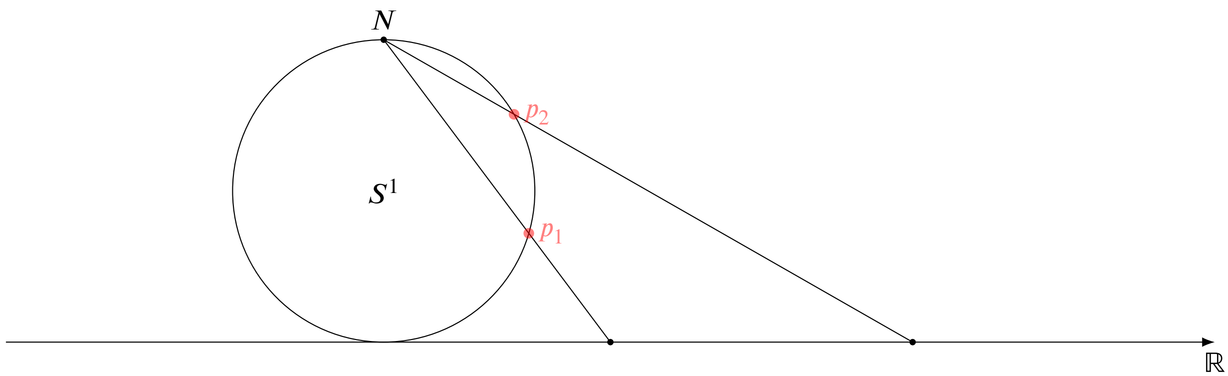


Figure 2: Stereographic projection of $S^1 \setminus \{N\}$ to the real line.

$$d(x, x_0) \leq d(x, x_{n_k}) + d(x_{n_k}, x_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so that $E_{n_k} \subset B(x, \epsilon) \subset U$, a contradiction.

(b) (X, d) is totally bounded.

Suppose not; then there exits $\epsilon > 0$ such that X cannot be covered by finitely many elements in $\{B(x, \epsilon)\}_{x \in X}$. Let x_1 be an arbitrary point in X; we can pick $x_2 \in X \setminus B(x_1, \epsilon)$. Similarly, we can pick $x_3 \in X \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon))$......Continuing this way, we obtain a sequence (x_n) where $x_n \in X \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon))$. Then (x_n) cannnot have a convergent subsequence, since $d(x_n, x_i) \geq \epsilon$ for $i = 1, \ldots, n-1$.

Let $\mathcal{O}$ be an open cover of X, and let $\delta > 0$ be a Lebesgue number for $\mathcal{O}$. Since X is totally bounded, $X \subset \bigcup_{i=1}^n B(x_i, \delta/3)$ for some $x_1, \ldots, x_n \in X$. Since each $B(x_i, \delta/3)$ has diameter less than δ, $B(x_i, \delta/3) \subset U_i \in \mathcal{O}, i = 1, \ldots, n$ for some $U_1, \ldots, U_n \in \mathcal{O}$. Then

$$X \subset \bigcup_{i=1}^n B(x_i, \delta/3) \subset \bigcup_{i=1}^n U_i,$$

□

and we are done. □

9.1 One-point Compactification

Fig. 2 is the stereographic projection of $S^1 \setminus \{N\}$ to the real line. $\mathbb{R}$ is homeomorphic to $S^1 \setminus \{N\}$. They are both not compact. However, if we add the "missing point" N, then the resulting space S^1 would be compact. Similarly, we can add a point to $\mathbb{R}$ to make it compact. One can think of this process as wrapping up the real line around the unit circle, and then joint the two ends of the line to form a "coherent" (compact) space.

Similarly, we can add a point N to the plane $\mathbb{R}^2$, to obtain a compact space $\mathbb{R}^2 \cup \{N\}$. We can think of this as wrapping up the plane around the punctured sphere $S^2 \setminus \{N\}$, and then add the final point N to glue the space together (Fig. 3).

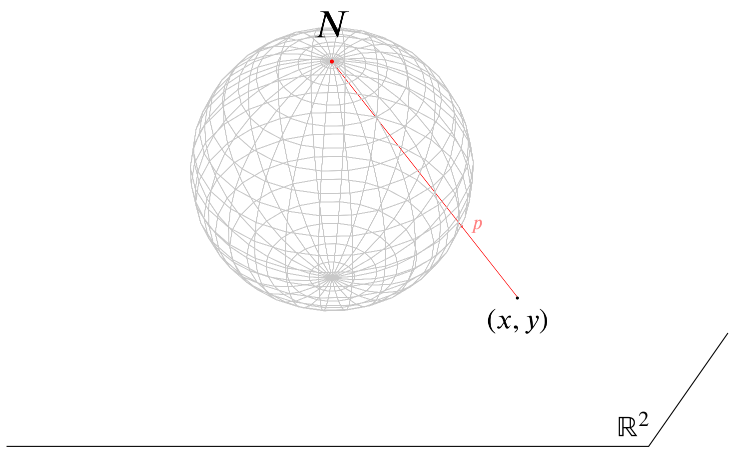


Figure 3: Stereographic projection of $S^2 \setminus \{N\}$ to the real plane.

Generally, suppose we have a non-compact space X at hand, and we would like to add a point to X to make a compact space $Y = X \cup \{N\}$. How should we define the topology on Y? Return to the above example of S^1, we see that for any neighborhood U of N, $S^1 \setminus U$ is homeomorphic to a closed interval, hence compact. This suggests that for our new space $Y = X \cup \{N\}$, any neighborhood of N should already cover "most portion of the space", namely only a compact subspace of X is left outside U. In this way, Y would easily be made compact.

Definition 9.24 Let $(X, \mathcal{T}_X)$ be a non-compact topological space. A *compactification* of X is a compact space Y such that $X \subset Y$ and $\bar{X} = Y$. If $Y \setminus X$ has only one point, then it is called a *one-point compactification* of X.

Construction 9.25. Let $(X, \mathcal{T}_X)$ be a non-compact space. Let N be a point not in X, and define the topology $\mathcal{T}_Y$ on $Y = X \cup \{N\}$ as

$$\mathcal{T}_Y = \mathcal{T}_X \cup \mathcal{O}_N = \{Y \setminus C : C \subset X \text{ compact }\}.$$

Lemma 9.26. *$\mathcal{T}_Y$ is indeed a topology on Y.* ✻

Proof. We verify that $\mathcal{T}_Y$ satisfies Definition 4.1.

1. $\varnothing, Y \in \mathcal{T}_Y$.

2. $$\begin{cases} U_1, U_2 \in \mathcal{T}_X & \Rightarrow \quad U_1 \cap U_2 \in \mathcal{T}_X; \\ Y \setminus C_1, Y \setminus C_2 \in \mathcal{O}_N & \Rightarrow \quad (Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) \in \mathcal{O}_N; \\ U \in \mathcal{T}_X, Y \setminus C \in \mathcal{O}_N & \Rightarrow \quad U \cap (Y \setminus C) = U \setminus C \in \mathcal{T}_X. \end{cases}$$

$$
3. \begin{cases} \{U_\lambda\} \subset \mathcal{T}_X & \Rightarrow \quad \bigcup U_\lambda \in \mathcal{T}_X; \\ \{Y \setminus C_\lambda\} \subset \mathcal{O}_N & \Rightarrow \quad \bigcup (Y \setminus C_\lambda) = Y \setminus (\bigcap C_\lambda) \in \mathcal{O}_N; \\ U \in \mathcal{T}_X, Y \setminus C \in \mathcal{O}_N & \Rightarrow \quad U \cup (Y \setminus C) = (Y \cap U) \cup (Y \setminus C) = Y \setminus (C \setminus U) \in \mathcal{O}_N \text{ by Lemma 2.1.} \end{cases}
$$

Proposition 9.27. *$(Y, \mathcal{T}_Y)$ is compact.* ♦

Proof. Any open cover $\mathcal{O}$ of $Y = X \cup \{N\}$ must contain $Y \setminus C \in \mathcal{O}_N$ for some compact subset C in X, in order to cover N, since $N \notin U$ for any $U \in \mathcal{T}_X$. Then since we are only left with a compact subset C, we can cover it by finitely many subcover. □

Observation 9.28. Note that, If X is already compact, then the construction above amounts to attach an isolated point $\{N\}$ to X. Indeed, since X is compact, $\{N\} = Y \setminus X \in \mathcal{O}_N$ is an open set containing N. X is also closed in Y in this case, namely $\bar{X} = X$, and we do not call such a compactification. If X is not compact, then X is open in $Y = X \cup \{N\}$, and N is a limit point of X, namely $(Y \setminus C) \cap X = X \setminus C \neq \varnothing$ for any $C \subset X$ compact, so that $\bar{X} = X \cup \{N\} = Y$. ⊙

Often, our space $(X, \mathcal{T}_X)$ will be Hausdorff, and we would like our one-point compactification $(Y, \mathcal{T}_Y)$ of X to be Hausdorff as well. Suppose X is Hausdorff and let $x, y \in Y = X \cup \{N\}$ be two distinct points. If $x, y \in X$, then we can find $U, V \in \mathcal{T}_X \subset \mathcal{T}_Y$ such that $x \in U, y \in V$, and $U \cap V = \varnothing$, because X is Hausdorff. If $x \in X$, while $y \in Y \setminus X = \{N\}$, then finding $x \in U \in \mathcal{T}_X$, $N \in Y \setminus C \in \mathcal{O}_N$ such that

$$U \cap (Y \setminus C) = \varnothing$$

is equivalent to finding a neighborhood U of x and a compact subset C such that

$$x \in U \subset C.$$

Motivated by this, we have the following notion:

> DEFINITION 9.29 X is said to be *locally compact* at $x \in X$ if there is some $C \subset X$ compact such that $x \in U \subset C$ for some neighborhood U of x. It is called *locally compact* if it is locally compact at every of its points.

Corollary 9.30. *If $(X, \mathcal{T})$ is locally compact and Hausdorff, then its one-point compactification $Y = X \cup \{N\}$ is Hausdorff as well.* ♠

We next do some exercises about the properties of locally compact spaces.

Exercise 9.31. Let $(X, \mathcal{T}_X)$ be Hausdorff. Then X is locally compact if and only if given $x \in X$, for any $x \in U \in \mathcal{T}_X$, there is $x \in V \in \mathcal{T}_X$ such that $x \in V \subset \bar{V} \subset U$ and $\bar{V}$ is compact. ◴

Proof. "⇐": This is trivial. Just take $C = \bar{V}$ and $U = V$ in Definition 9.29.

"⇒": Let $x \in U \in \mathcal{T}_X$. Take one-point compactification $Y = X \cup \{N\}$ of X. Since $C = Y \setminus U$ is closed in Y, it is compact by Proposition 9.4. Thus as in the proof of Proposition 9.5, we can find

$x \in V \in \mathcal{T}_X$, $C \subset W \in \mathcal{T}_Y$ such that $V \cap W = \varnothing$. Then $V \subset Y \setminus W$, which is closed, so that $\bar{V} \subset Y \setminus W$. Since $C = Y \setminus U \subset W$, we have $U = Y \setminus (Y \setminus U) \supset Y \setminus W \supset \bar{V} \supset V$. What's more, $\bar{V}$ is compact, by Proposition 9.4 again. □

10 Countability Axioms

DEFINITION 10.1 Let X be a topological space and let $\mathcal{N}_x$ be the set of all neighborhoods of x. X is said to have a *countable basis at* x if there is a countable $\mathcal{N}_0 \subset \mathcal{N}_x$ and a function $f : \mathcal{N}_x \to \mathcal{N}_0$ such that

$$f(N) \subset N$$

for all $N \in \mathcal{N}_x$. We say X is *first countable* if it has a countable basis at every $x \in X$.

- A metric space is first countable: $\{B(x, 1/n)\}_{n\in\mathbb{N}}$ is a countable basis at x.
- $\mathbb{R}_\ell$ is first countable: $\{[x, x + 1/n)\}_{n\in\mathbb{N}}$ is a countable basis at x.

The significance of a first countable space lies in the fact that sequences are enough to characterize limit points and continuous functions.

Proposition 10.2. *Let X be a topological space.*

(a) Let $A \subset X$. If there is a sequence (x_n) in A such that $x_n \to x \in X$, then $x \in \bar{A}$. The converse holds if X is first countable.

(b) Let $f : X \to Y$. If f is continuous, then $x_n \to x \Rightarrow f(x_n) \to f(x)$. The converse holds if X is first countable. ◆

Proof. Suppose X is first countable.

(a) Let $x \in \bar{A}$, and let $\mathcal{N}_0 = \{N_1, N_2, \dots\}$. Pick

$$x_n \in \left(\bigcap_{i=1}^{n} N_i\right) \cap A$$

for each $n \in \mathbb{N}$. Then it is easy to see that $x_n \to x$: for every $N \in \mathcal{N}_x$,

$$f(N) = N_k \subset N$$

for some $k \in \mathbb{N}$. Then

$$x_n \in \bigcap_{i=1}^{n} N_i \subset N$$

for all $n \geq k$.

(b) We prove $f(\bar{A}) \subset \overline{f(A)}$. Let $x \in \bar{A}$. Then by (a), there is a sequence (x_n) in A such that $x_n \to x$. By assumption, $f(x_n) \to f(x)$, so that $f(x) \in \overline{f(A)}$. □

DEFINITION 10.3 If a space X has a countable basis, then we say X is *second countable*.

- $\mathbb{R}$ is second countable: $\{(a,b)\}_{a,b\in\mathbb{Q}}$ is a countable basis for $\mathbb{R}$. Similarly, $\mathbb{R}^n$ is second countable.
- If X is an uncountable set, the discrete topology on X does not have a countable basis. The discrete topology can be generated by the discrete metric, so this shows that not every metric space is second countable.
- $\mathbb{R}_\ell$ is not second countable. The lower limit topology is too fine on $\mathbb{R}$ such that it is not possible for a countable basis to generate this topology. To see this, let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Then for every $x \in \mathbb{R}$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset [x, x+1)$. Further, $B_x \neq B_y$ for $x \neq y$. This shows that the function $x \mapsto B_x$ is injective, so that $\mathcal{B}$ has cardinality at least as large as $\mathbb{R}$, so that $\mathcal{B}$ can not be countable.

DEFINITION 10.4 $A \subset X$ is said to be *dense* in X if $\bar{A} = X$. If X has a countable dense subset, then X is said to be *separable*.

Proposition 10.5. *Let X be a topological space.*

(a) If X is second countable, then it is Lindelöf.

(b) If X is second countable, then it is separable. ◆

Proof. Suppose X is second countable, and let $\mathcal{B} = \{B_n\}$ be a countable basis.

(a) Let $\mathcal{A}$ be an open cover of X. For each B_n, choose $A_n \in \mathcal{A}$ such that $B_n \subset A_n$, if this is possible. Then $\mathcal{A}' = \{A_n\}$ is countable. We claim that it covers X. For each $x \in X$, we have $x \in A$ for some $A \in \mathcal{A}$. Since A is open, we have $x \in B_n \subset A$ for some $B_n \in \mathcal{B}$. Then $x \in B_n \subset A_n$. This proves that $\mathcal{A}' = \{A_n\}$ indeed covers X.

(b) Choose $x_n \in B_n$ for each $B_n \in \mathcal{B}$. Then $\{x_n\}$ is dense in X. □

Exercise 10.6. A subspace of a first (second) countable space is first (second) countable, and a countable product of first (second) countable spaces is second-countable. ◴

Proof. Immediate. □

Exercise 10.7. Let A be an uncountable subset of a second countable space $(X, \sigma(\mathcal{B}))$. Show that uncountably many points of A are limit points of A. ◴

Proof. Let A_0 be the subset of A that are limit points of A, and suppose it is countable. Then $A \setminus A_0$ is uncountable. For every $x \in A \setminus A_0$, there is a basis element B_x such that $B_x \cap A = \{x\}$. Furthermore, for $x \neq y \in A \setminus A_0$ we have $B_x \neq B_y$ since $B_x \cap A \neq B_y \cap A$. The map $x \mapsto B_x$ is thus injective from $A \setminus A_0$ to $\mathcal{B}$, so that $\mathcal{B}$ cannot be countable. □

Exercise 10.8. Show that every compact metrizable space is second countable. ◴

Proof. Let (X, d) be a compact metric space. For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be the collection of those finitely many elements in $\{B(x, 1/n)\}_{x \in X}$ that cover X. Then

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$$

is a countable basis of X. □

Exercise 10.9. Show that $\mathbb{R}_\ell$ is Lindelöf. ◴

Proof. Let $\mathcal{O} = \{[a, b)\}$ be an open cover of $\mathbb{R}$ by basis elements. For any $x \in \mathbb{R}$, define

$$C_x = \{y \geq x \mid [x, y] \text{ can be countably covered}\}.$$

Then it must be the case that $\sup C_x = \infty$. For suppose $\sup C_x = z < \infty$. Then there is $[a, b) \in \mathcal{O}$ such that $z \in [a, b)$. Pick $y \in C_x \cap [a, b)$, so that $[x, y]$ is countably covered, and pick $z' \in (z, b)$. Then $[x, z'] = [x, y] \cup [y, z']$ can be countably covered, since $[y, z'] \subset [a, b)$. This shows $z' \in C_x$, contradicting to the fact that $z = \sup C_x$. Thus any closed interval in $\mathbb{R}$ can be countably covered, and so is $\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n]$. □

From Proposition 10.5, we see that Lindelöf and separability is weaker than the second countability axiom. $\mathbb{R}_\ell$ is such an example that is both Lindelöf and separable (rational numbers are dense in $\mathbb{R}_\ell$) but not second countable. However, for metric spaces these three are equivalent.

Exercise 10.10. Let (X, d) be a metric space.

(a) If X is Lindelöf, then it is second countable.

(b) If X is separable, then it is second countable. ◴

Proof. (a) For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be the collection of those countably many elements in $\{B(x, 1/n)\}_{x \in X}$ that cover X. Then

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$$

is a countable basis of X.

(b) Let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense subset in X. Then

$$\left\{B(x_n, 1/m)\right\}_{n,m \in \mathbb{N}}$$

is a countable basis of X. □

Corollary 10.11. $\mathbb{R}_\ell$ *is not metrizable.* ♠

Exercise 10.12. Show that $\mathbb{R}_\ell^2$ is *not* Lindelöf. ◴

Proof. Let

$$L = \{(x, -x) \mid x \in \mathbb{R}\}.$$

Then L is closed in $\mathbb{R}^2_\ell$. Indeed, for any $(x, y) \in \mathbb{R}^2 \setminus L$, we have $(x, y) \in [x, a) \times [y, b) \subset \mathbb{R}^2 \setminus L$ for some $a, b \in \mathbb{R}$, so that $\mathbb{R}^2 \setminus L$ is open. Now consider the open cover of $\mathbb{R}^2$ by

$$\mathcal{O} = \left\{\mathbb{R}^2 \setminus L\right\} \cup \{[x, x + a) \times [-x, -x + a) \mid x \in \mathbb{R}\}$$

for some $a > 0$. Since $[x, x + a) \times [-x, -x + a) \cap L = \{(x, -x)\}$ for each $x \in \mathbb{R}$, remove any $[x, x + a) \times [-x, -x + a)$ would result in $\{(x, -x)\} \in \mathbb{R}^2$ not being covered. □

11 Separation Axioms

DEFINITION 11.1 Let $(X, \mathcal{T})$ be a topological space. Given two sets A and B in X, we say A and B can be separated in the topology if there exist $U, V \in \mathcal{T}$ such that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

1. Suppose one point sets are closed in $(X, \mathcal{T})$. Then X is said to be *regular* if for any closed set C in X, any $x \notin C$ can be separated in the topology from C.

2. Suppose one point sets are closed in $(X, \mathcal{T})$. Then X is said to be *normal* if every pair of disjoint closed sets can be separated in the topology.

Proposition 11.2. *Let $(X, \mathcal{T})$ be a topological space where one-point sets are closed.*

1. *X is regular if and only if given $x \in X$ and a neighborhood U of x, there is an open set V such that $x \in V$ and $\bar{V} \subset U$.*

2. *X is normal if and only if given a closed set A and an open set U such that $A \subset U$, there is an open set V such that $A \subset V$ and $\bar{V} \subset U$.* ◆

Proof. 1. Suppose first that X is regular, and let $x \in U$. Then $x \notin X \setminus U$, so that there is $V, W \in \mathcal{T}$ such that $x \in V$, $X \setminus U \subset W$, and $V \cap W = \varnothing$. We then have $V \subset X \setminus W \subset U$, and since $X \setminus W$ is closed, we have $\bar{V} \subset X \setminus W \subset U$, as desired.

To prove the converse, let C be closed in X. Then for every $x \in X \setminus C$, there is open set V such that $x \in V \subset \bar{V} \subset X \setminus C$. Observe that $X \setminus \bar{V} \supset C$, and $V \cap (X \setminus \bar{V}) = \varnothing$. x and C are thus separated by V and $X \setminus \bar{V}$.

2. Suppose first that X is normal, and let $A \subset U$. Then $A \cap (X \setminus U) = \varnothing$, so that there is $V, W \in \mathcal{T}$ such that $A \subset V$, $X \setminus U \subset W$, and $V \cap W = \varnothing$. We then have $V \subset X \setminus W \subset U$, and since $X \setminus W$ is closed, we have $\bar{V} \subset X \setminus W \subset U$, as desired.

To prove the converse, let A, B be closed sets in X such that $A \cap B = \varnothing$. Then $A \subset X \setminus B$, which is open, so that by assumption there is V open such that $A \subset V \subset \bar{V} \subset X \setminus B$. Observe that $X \setminus \bar{V} \supset B$, and $V \cap (X \setminus \bar{V}) = \varnothing$. A and B are thus separated by V and $X \setminus \bar{V}$. □

Example 11.3. We define a topology on $\mathbb{R}$ to make it fail to be regular, but still Hausdorff. To do this, observe that 0 and $K = \{1/n : n \in \mathbb{N}\}$ are very "close": 0 is a limit point of K in the usual topology. If we can make K closed in our new topology, then the point 0, which is not in K, would be very very close to the closed set K, and they may not be separated provided our new topology is not too fine. Now, how to make K closed? Easy, just declare $(a, b) \setminus K$ to be open! Formally, we let $\mathcal{B}_K$ be the basis on $\mathbb{R}$ consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$. The topology generated by the basis $\mathcal{B}_K$ is denoted by $\mathcal{T}_K$, and we write $\mathbb{R}_K = (\mathbb{R}, \mathcal{T}_K)$. K is closed in $\mathbb{R}_K$, and $\mathbb{R}_K$ is easily seen to be Hausdorff. Suppose now we want to separate 0 and K. Then for neighborhood of 0, we must choose sets of the form $(a, b) \setminus K$. Without loss of generality we assume it is $(-\epsilon, \epsilon) \setminus K$, where $0 < \epsilon < 1$. For an open set that contains K we must choose sets of the form (a, b), and without loss of generality we assume it is $(0, r)$ for some $r > 1$. Now $0 \in (-\epsilon, \epsilon) \setminus K$, $K \subset (0, r)$, but obviously the two open sets cannot be disjoint. Indeed, by Lemma 2.2 we have $\big((-\epsilon, \epsilon) \setminus K\big) \cap (0, r) = \big[(-\epsilon, \epsilon) \cap (0, r)\big] \setminus K = (0, \epsilon) \setminus K$, and the last set is obviously not empty. ◁

Example 11.4. Every metric space is normal. To see this, let (X, d) be a metric space, and A and B be two disjoint closed sets in X. Since $X \setminus B$ is open, for every $x \in A$ there is an open ball $B(x, \epsilon_x)$ such that $x \in B(x, \epsilon_x) \subset X \setminus B$, and similarly for every $y \in B$ there is $B(y, \epsilon_y)$ such that $y \in B(y, \epsilon_y) \subset X \setminus A$. Then

$$U = \bigcup_{x \in A} B(x, \epsilon_x/2) \quad \text{and} \quad V = \bigcup_{y \in B} B(y, \epsilon_y/2)$$

are open sets containing A and B respectively. To see $U \cap V = \emptyset$, let $z \in U \cap V$. Then $z \in B(x, \epsilon_x/2) \cap B(y, \epsilon_y/2)$ for some $x \in A$ and $y \in B$. Then

$$d(x, y) \le d(x, z) + d(z, y) < \epsilon_x/2 + \epsilon_y/2 \le \max\{\epsilon_x, \epsilon_y\},$$

a contradiction to our selection of open balls. ◁

Exercise 11.5. Let $f, g : X \to Y$ be continuous, where Y is Hausdorff. Show that $A = \{x \in X | f(x) = g(x)\}$ is closed in X. ◴

Proof. We prove $X \setminus A = \{x \in X | f(x) \neq g(x)\}$ is open in X. Let $x \in X \setminus A$, so that $f(x) \neq g(x)$. Then since Y is Hausdorff, there are open sets U, V in Y such that $f(x) \in U$, $g(x) \in V$, and $U \cap V = \emptyset$. We then have $x \in f^{-1}(U)$ as well as $x \in g^{-1}(V)$, so that $x \in f^{-1}(U) \cap g^{-1}(V)$. Given any $y \in f^{-1}(U) \cap g^{-1}(V)$, we have $f(y) \in U$ and $g(y) \in V$, and since $U \cap V = \emptyset$, we have $f(y) \neq g(y)$, so that $y \in X \setminus A$. This proves $x \in \big(f^{-1}(U) \cap g^{-1}(V)\big) \subset X \setminus A$, so $X \setminus A$ is open. □

Exercise 11.6. Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y. ◴

Proof. Let A, B be closed sets in Y such that $A \cap B = \emptyset$. Then $p^{-1}(A)$ and $p^{-1}(B)$ are closed in X, and $p^{-1}(A) \cap p^{-1}(B) = \emptyset$. By normality of X, there exist U, V open in X such that $p^{-1}(A) \subset U$, $p^{-1}(B) \subset V$, and $U \cap V = \emptyset$. Now to find disjoint open subsets in Y that contain A and B respectively, we first take the complements $X \setminus U$ and $X \setminus V$, which are closed, and send them to Y via p, obtaining two closed sets $p(X \setminus U)$ and $p(X \setminus V)$ in Y, then finally take the complements $Y \setminus p(X \setminus U)$ and

$Y \setminus p(X \setminus V)$. These two open sets in Y are disjoint due to the surjectivity of p. Indeed, we have

$$\begin{aligned}\big(Y \setminus p(X \setminus U)\big) \cap \big(Y \setminus p(X \setminus V)\big) &= Y \setminus \big[p(X \setminus U) \cup p(X \setminus V)\big] \\ &= Y \setminus p\,\big[(X \setminus U) \cup (X \setminus V)\big] \\ &= Y \setminus p\big(X \setminus (U \cap V)\big) \\ &= Y \setminus p(X) \\ &= Y \setminus Y = \varnothing.\end{aligned}$$

Now we claim $A \subset Y \setminus p(X \setminus U)$. From $p^{-1}(A) \subset U$, we have $X \setminus U \subset X \setminus p^{-1}(A) = p^{-1}(Y \setminus A)$. Then

$$p(X \setminus U) \subset p\big(p^{-1}\big(Y \setminus A\big)\big) \subset Y \setminus A,$$

so that $Y \setminus p(X \setminus U) \supset A$. The proof for $Y \setminus p(X \setminus V) \supset B$ is similar. This completes the proof that Y is normal. □

Exercise 11.7. Let $f : X \to Y$ be a closed continuous surjective map such that $f^{-1}(\{y\})$ is compact for each $y \in Y$. ◴

(a) Show that if X is Hausdorff, then so is Y.

Recall how we proved that every compact set in a Hausdorff space is closed in Proposition 9.5. Its proof can be used to show that Hausdorff space is "regular" and "normal" with respect to *compact* sets.

Lemma 11.8. *In a Hausdorff space X, any compact set A can be separated from points not in A.* ✻

Proof. Fix $x_0 \in X \setminus A$. Then for every $y \in A$, there are open sets U_y and V_y such that $x_0 \in U_y$, $y \in V_y$, and $U_y \cap V_y = \varnothing$. $\{V_y\}_{y \in A}$ is an open cover of A; by compactness of A, $A \subset \bigcup_{i=1}^n V_{y_i} =:$ V for some finite number of points $y_1, \dots, y_n$ in A. Denote $U := \left(\bigcap_{i=1}^n U_{y_i}\right)$. Then $x \in U$, $A \subset V$, and $U \cap V = \varnothing$, which is desired. □

Lemma 11.9. *In a Hausdorff space X, every pair of disjoint compact sets A and B can be separated.* ✻

Proof. By Lemma 11.8, for every $x \in A$, there is open sets U_x and V_x such that $x \in U_x$, $B \subset V_x$, and $U_x \cap V_x = \varnothing$. By compactness of A, the open cover $\{U_x\}_{x \in A}$ of A has a finite subcover so that $A \subset U_{x_1} \cup \cdots \cup U_{x_n}$ for some finite set of points $x_1, \dots, x_n$ in A. Then $U := U_{x_1} \cup \cdots \cup U_{x_n}$ and $V := V_{x_1} \cap \cdots \cap V_{x_n}$ are disjoint open sets that contain A and B respectively. □

Proof (Proof of (a)). Let $y_1 \neq y_2 \in Y$. Then $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are closed disjoint sets in X. We would not have proceeded as in Exercise 11.6 to find open sets U and V in X such that $p^{-1}(\{y_1\}) \subset U$, $p^{-1}(\{y_2\}) \subset V$, and $U \cap V = \varnothing$, since we do not assume X to be normal, but merely Hausdorff. Now our extra condition that "$f^{-1}(\{y\})$ is compact for each $y \in Y$" comes to rescue, in light of Lemma 11.9. □

(b) Show that if X is regular, then so is Y.

Proof. Given closed set C in Y and $y \notin C$, we would like to find open sets U and V in X such that $p^{-1}(\{y\}) \subset U$, $p^{-1}(C) \subset V$, and $U \cap V = \varnothing$, so that we can proceed as in the proof of Exercise 11.6. The proof is in the same spirit as in Lemma 11.9. For every $x \in p^{-1}(\{y\})$, by regularity of X we can find open sets U_x and V_x in X such that $x \in U_x$, $p^{-1}(C) \subset V_x$, and $U_x \cap V_x = \varnothing$. $\{U_x\}_{x \in p^{-1}(\{y\})}$ is an open cover of $p^{-1}(\{y\})$; by compactness of $p^{-1}(\{y\})$ there are $x_1, \dots, x_n$ in $p^{-1}(\{y\})$ such that $p^{-1}(\{y\}) \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Then $U := U_{x_1} \cup \cdots \cup U_{x_n}$ and $V := V_{x_1} \cap \cdots \cap V_{x_n}$ are the desired open sets we want to find. □

(c) Show that if X is locally compact, then so is Y.

Proof. Let $y \in Y$ be arbitrary. Consider $p^{-1}(\{y\})$. Since X is locally compact, for every $x \in p^{-1}(\{y\})$ there are some open set U_x and compact set C_x such that $x \in U_x \subset C_x$. $\{U_x\}_{x \in p^{-1}(\{y\})}$ is an open cover of $p^{-1}(\{y\})$; by compactness of $p^{-1}(\{y\})$ there are $x_1, \dots, x_n$ in $p^{-1}(\{y\})$ such that $p^{-1}(\{y\}) \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Let $U := U_{x_1} \cup \cdots \cup U_{x_n}$ and $C := C_{x_1} \cup \cdots \cup C_{x_n}$. Then $p^{-1}(\{y\}) \subset U \subset C$, where U is open and C is compact. From this we have

$$X \setminus U \supset X \setminus C$$
$$\Downarrow$$
$$p(X \setminus U) \supset p(X \setminus C)$$
$$\Downarrow$$
$$Y \setminus p(X \setminus U) \subset Y \setminus p(X \setminus C) = Y \setminus (Y \setminus p(C)) = p(C).$$

We have $y \in Y \setminus p(X \setminus U) \subset p(C)$, where $Y \setminus p(X \setminus U)$ is open and $p(C)$ is compact. This completes the proof that Y is locally compact. □

From Lemma 11.9, if a Hausdorff space is compact, then it is automatically normal, since every closed set in a compact space is compact.

Proposition 11.10. *Every compact Hausdorff space is normal.* ◆

We can use the similar idea in the proof of Lemma 11.9 to obtain the following result.

Proposition 11.11. *Every Lindelöf regular space is normal.* ◆

Proof. Suppose X is regular and Lindelöf, and let A and B be two disjoint closed sets in X. By regularity of X, for each $x \in A$ we can choose a neighborhood U_x of x such that $x \in U_x \subset \bar{U}_x \subset X \setminus B$. Similarly, for each $y \in B$ there is V_y such that $y \in V_y \subset \bar{V}_y \subset X \setminus A$. $\{U_x\}_{x \in A}$ and $\{V_y\}_{y \in B}$ are covers of A and B respectively; since A and B are themselves Lindelöf (because they are closed in X), we have $A \subset \bigcup_{n=1}^{\infty} U_n$ and $B \subset \bigcup_{n=1}^{\infty} V_n$ for some countable subcovers.

Now define $U_n' = U_n \setminus \bigcup_{i=1}^{n} \bar{V}_n$ and $V_n' = V_n \setminus \bigcup_{i=1}^{n} \bar{U}_n$. Then for $U = \bigcup_{n=1}^{\infty} U_n'$ and $V = \bigcup_{n=1}^{\infty} V_n'$ we have $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$, as desired. □

12 Urysohn Lemma and Its Applications

Theorem 12.1 (Urysohn Lemma). *Let X be a normal space, and let A and B be two disjoint closed sets in X. Then there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 1$ and $f(B) = 0$.*$\diamond$

Proof. We shall find a family of open sets around A and index them by dyadic rationals in $[0,1]$, and define our continuous function using the index.

Let $U_0 = X \setminus B$. Since $A \subset X \setminus B$, we have by normality an open set $U_{1/2}$ such that

$$A \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_0.$$

Applying the above process again to the closed sets A and $\bar{U}_{1/2}$, we can find $U_{3/4}$ and $U_{1/4}$ such that

$$A \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_0.$$

We index the open sets closer to A using dyadic rationals that are closer to 1, because we want $f(A)$ to be equal to 1, and points closer to A to have higher values, while for points that are further and further away from A we want to assign them smaller and smaller values so that eventually $f(x) = 0$ for all $x \in B$. Continuing our construction, we obtain a family of open sets $\{U_p\}$ indexed by dyadic rationals $k/2^n$ in $[0,1]$ such that $\bar{U}_p \subset U_q$ for all $p > q$.

Now define $f : X \to [0,1]$ by $f(x) = \sup\{p \mid x \in U_p\}$ for $x \in X \setminus B$, and $f(x) = 0$ for $x \in B$. To prove f is continuous, by Exercise 7.9 we only need to verify that $\{f(x) > a\}$ and $\{f(x) < a\}$ are open in X. To make matters more clear, let us denote the set $\{p \mid x \in U_p\}$ by A_x, so that $p \in A_x$ if and only if $x \in U_p$. We want to show $f(x) = \sup A_x$ is continuous. Let $a \in (0,1)$ be given, and let $p > a$. For any $x \in U_p$, we have $p \in A_x$, so that $a < p \leq \sup A_x = f(x)$. This proves $U_p \subset \{f(x) > a\}$ for any $p > a$, so we have $\bigcup_{p>a} U_p \subset \{f(x) > a\}$. Conversely, if $a < \sup A_x = f(x)$, then there is some $p \in A_x$ such that $a < p \leq \sup A_x$, so that $x \in U_p$. This proves $\{f(x) > a\} \subset \bigcup_{p>a} U_p$. Thus we have $\{f(x) > a\} = \bigcup_{p>a} U_p$.

Next, let $p < a$. If $x \notin \bar{U}_p$, then $x \notin U_{p'}$ for all $p' > p$, so that $p' \notin A_x$ for all $p' > p$. We can then derive that $f(x) = \sup A_x \leq p < a$. This shows $X \setminus \bar{U}_p \subset \{f(x) < a\}$ for all $p < a$, so that $\bigcup_{p<a} X \setminus \bar{U}_p \subset \{f(x) < a\}$. Conversely, if $x \in X$ is such that $f(x) < a$, then we can find two dyadic numbers p and p' such that $f(x) < p' < p < a$. We have $x \notin U_p$, since otherwise $p \in A_x$ and thus $p \leq \sup A_x = f(x)$, a contradiction. By our construction we have $\bar{U}_{p'} \subset U_p$, so $x \notin \bar{U}_{p'}$ as well. This proves $\{f(x) < a\} \subset \bigcup_{p<a} X \setminus \bar{U}_p$. We thus have $\{f(x) < a\} = \bigcup_{p<a} X \setminus \bar{U}_p$. We have proved that both $\{f(x) < a\}$ and $\{f(x) > a\}$ can be written as unions of open sets in X, so they are open. This completes the proof that our f is indeed continuous. $\square$

There is no speciality of the interval $[0,1]$ in the statement of the theorem, and we can replace it by an arbitrary closed interval $[a,b]$.

Theorem 12.2 (Tietze Extension Theorem). *Let X be a normal space and let A be a closed subset of X. Any continuous function $f : A \to [a,b]$ can be extended to a continuous function $g : X \to [a,b]$.*$\diamond$

Proof. Given a continuous function f from A to $[-r,r]$, we can use Urysohn lemma to construct a continuous function g on X such that

(1) $|g| \leq \frac{r}{3}$ for all $x \in X$;

(2) $|f - g| \leq \dfrac{2r}{3}$ for all $a \in A$.

To do this, consider $I_1 = [-r, -r/3], I_2 = [-r/3, r/3], I_3 = [r/3, r]$, and let $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$. B and C are closed and disjoint in X, so by Urysohn Lemma there exists continuous function $g : X \to [-r/3, r/3]$ such that $g(B) = -r/3$ and $g(C) = r/3$. It is easy to see that g satisfies (1) and (2).

Now we prove the theorem. Without loss of generality, let $f : A \to [-1, 1]$ be a continuous function on A that takes values in $[-1, 1]$. Apply the above procedure to f, we obtain a continuous function g_1 on X such that

(1) $|g_1| \leq \dfrac{1}{3}$ for all $x \in X$;

(2) $|f - g_1| \leq \dfrac{2}{3}$ for all $a \in A$.

Apply the same construction to the function $f - g_1 : A \to [-2/3, 2/3]$, we obtain a continuous function g_2 on X such that

(1) $|g_2| \leq \dfrac{1}{3}\left(\dfrac{2}{3}\right)$ for all $x \in X$;

(2) $|f - g_1 - g_2| \leq \dfrac{2}{3}\left(\dfrac{2}{3}\right)$ for all $a \in A$.

By induction, we get a sequence of continuous functions $\{g_n\}_{n\in\mathbb{N}}$ on X such that for each $n \in \mathbb{N}$

(1) $|g_n| \leq \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{n-1}$ for all $x \in X$;

(2) $|f - \displaystyle\sum_{i=1}^{n} g_i| \leq \left(\dfrac{2}{3}\right)^{n}$ for all $a \in A$.

Let $g = \sum_{n=1}^{\infty} g_n$. We show g is the desired extension of f. First, by comparison test g indeed converges on X: we have

$$|g| = \left|\sum_{n=1}^{\infty} g_n\right| \leq \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 1.$$

From (2) we have

$$\lim_{n\to\infty} |f - \sum_{i=1}^{n} g_i| = |f - g| = 0$$

on A, so that $f(a) = g(a)$ for all $a \in A$. It is also clear that $\sum_{i=1}^{n} g_i$ converges uniformly to g:

$$|g - g_n| = \left|\sum_{i=n+1}^{\infty} g_i(x)\right| \leq \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^{n}.$$

Continuity of g follows. This completes the proof that $g : X \to [-1, 1]$ is the desired extension of $f : A \to [-1, 1]$. □

Corollary 12.3. *Let X be a normal space and let A be a closed subset of X. Any continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $g : X \to \mathbb{R}$.* ♠

Proof. Without loss of generality, let $f : A \to (-1, 1)$ be a continuous function on A that takes values in $(-1, 1) \subset [-1, 1]$. By the Tietze Extension Theorem, we can extend f to a continuous function $g : X \to [-1, 1]$. What if there is some $x \in X$ such that $g(x) = 1$ or $g(x) = -1$? Kill them! Note that $A \subset g^{-1}(-1, 1)$ and $E = g^{-1}\{-1, 1\}$ are disjoint in X, so by Urysohn Lemma there is continuous function $\lambda : X \to [-1, 1]$ such that

$$\lambda(A) = 1 \quad \text{and} \quad \lambda(E) = 0.$$

We use λ to "kill" the set of points in X on which g has values in $\{-1, 1\}$. So let g' be defined by

$$g'(x) = \lambda(x)g(x)$$

for all $x \in X$. It is easy to see that $g(a) = \lambda(a)g(a) = 1 \cdot g(a) = g(a) = f(a)$ for $a \in A$. Also, when $x \in E$, we have $g'(x) = \lambda(x)g(x) = 0 \cdot g(x) = 0$ and for $x \notin E$ we have $g'(x) = \lambda(x)g(x) \leq 1 \cdot g(x) < 1$. Thus $g' : X \to (-1, 1)$ is our desired extension of $f : A \to (-1, 1)$. □

Exercise 12.4. The Tietze Extension Theorem implies the Urysohn Lemma. ◴

Proof. Let A and B be two closed and disjoint subsets in a normal space X. Define $f : A \cup B \to [0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \in B. \end{cases}$$

Since $A \cap B = \emptyset$, f is continuous. Apply the Tietze Extension Theorem, we get a continuous extension $g : X \to [0, 1]$ of f such that $g(x) = f(x)$ for $x \in A \cup B$. Thus $g(x) = 1$ for $x \in A$ and $g(x) = 0$ for $x \in B$, as desired. □

Our next application of the Urysohn Lemma is the embedding of compact manifolds in Euclidean space $\mathbb{R}^N$.

DEFINITION 12.5 $f : X \to Y$ is an *embedding* of X into Y if $f : X \to f(X)$ is a homeomorphism. Namely, f is injective and continuous, and $f^{-1} : f(X) \to X$ is also continuous.

DEFINITION 12.6 An *m-manifold* M is a second countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^m$.

By "locally homeomorphic to $\mathbb{R}^m$" we mean each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^m$. The most familiar example of a manifold is probably the place you are standing on: the surface of our earth (here we approximate the surface of earth by the unit ball $S^2 = \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$) This is the manifold that we can directly feel in reality: the surface of earth lies in a three dimensional space, but looking around, we feel like we are living on a giant plane, and the surface of earth can thus be seen as a 2-manifold.

The requirement that a manifold be second countable and Hausdorff is equally important with the local homeomorphism assumption. They together ensure that a manifold can indeed be embedded in Euclidean space. In other word, the local homeomorphism assumption alone does not guarantee that a space can have all the crucial topological properties of the Euclidean space.

The *support* of a real valued function ϕ on X is $\overline{\{x \in X : \phi(x) \neq 0\}}$. We denote the support of a function ϕ by suppϕ.

Definition 12.7 Let $\{U_1, \ldots, U_n\}$ be a finite open cover of X. An indexed family of continuous functions $\phi_i : X \to [0,1], i = 1, \ldots, n$ is said to be a *partition of unity* subordinated to $\{U_i\}$ if

(1) $\text{supp}\phi_i \subset U_i$ for each i;

(2) $\sum_{i=1}^n \phi_i =: \Phi \equiv 1$ on X.

Lemma 12.8. *Let $\{U_1, \ldots, U_n\}$ be a finite open cover of of X. If X is normal, then there exists a partition of unity subordinated to $\{U_i\}$.* ✻

Proof. First, we prove that there is an open cover $\{V_1, \ldots, V_n\}$ of X such that $\bar{V}_i \subset U_i$ for each i. Since $X = U_1 \cup U_2 \cup \cdots \cup U_n$, we have $A_1 = X \setminus (U_2 \cup \cdots \cup U_n) \subset U_1$. By normality, there is open set V_1 such that $A_1 \subset V_1$ and $\bar{V}_1 \subset U_1$. Then $\{V_1, U_2, \ldots, U_n\}$ covers X. Similarly, $A_2 = X \setminus (V_1 \cup \cdots \cup U_3 \cup \cdots \cup U_n) \subset U_2$, so that there is open set V_2 such that $A_2 \subset V_2$ and $\bar{V}_2 \subset U_2$. Then $\{V_1, V_2, U_3, \ldots, U_n\}$ covers X. In general, after we find $V_1, \ldots, V_{k-1}$ such that $\{V_1, \ldots, V_{k-1}, U_k, \ldots, U_n\}$ covers X, we have $A_k = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n) \subset U_k$, so that there is open set V_k such that $A_k \subset V_k$ and $\bar{V}_k \subset U_k$, and $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ covers X. At $k = n$ we obtain an open cover $\{V_1, \ldots, V_n\}$ of X such that $\bar{V}_i \subset U_i$ for each i, as desired.

We can similarly find open cover $\{W_1, \ldots, W_n\}$ such that $\bar{W}_i \subset V_i$ for each i. We can now apply Urysohn Lemma to obtain a continuous function $\psi_i : X \to [0,1]$ for each i such that $\psi_i(\bar{W}_i) = 1$ and $\psi_i(X \setminus V_i) = 0$. Then since $\{x \in X : \psi_i(x) \neq 0\} \subset V_i$, we have

$$\text{supp}\psi_i = \overline{\{x \in X : \psi_i(x) \neq 0\}} \subset \bar{V}_i \subset U_i.$$

Given $x \in X$, we have $x \in W_i$ for some i, so that $\Psi(x) = \sum_{i=1}^n \psi_i(x) > 1$. The functions

$$\phi_i(x) = \frac{\psi_i(x)}{\Psi(x)}$$

for $i = 1, \ldots, n$ constitute the desired partition of unity. □

Lemma 12.9. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.* ✻

Proof. A closed in $X \Rightarrow A$ compact in $X \Rightarrow f(A)$ compact in $Y \Rightarrow f(A)$ closed in Y. This proved that $f^{-1} : Y \to X$ is continuous. □

Theorem 12.10 (Embedding of Manifolds). *If M is a compact m-manifold, then it can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.* ◇

Proof. For every $x \in M$ we can find a neighborhood U_x of x such that U_x is homeomorphic to an open set in $\mathbb{R}^m$. The collection $\{U_x\}_{x \in M}$ covers M, so by compactness there is a subset $\{U_1, \ldots, U_n\}$ of $\{U_x\}_{x \in M}$ that also covers M. We thus have n continuous functions $f_i : U_i \to \mathbb{R}^m$ at our disposal, where each f_i is an embedding of U_i into $\mathbb{R}^m$. Because M is compact and Hausdorff, M is normal, so that we have partition of unity $\phi_1, \ldots, \phi_n$ subordinated to $\{U_i\}$. Define $f_i' : M \to \mathbb{R}^m$ for each

$i = 1, \dots, n$ by

$$f_i'(x) = \begin{cases} \phi_i(x) \cdot f_i(x) & x \in U_i; \\ \mathbf{0} & x \in M \setminus \mathrm{supp}\phi_i. \end{cases}$$

For $x \in U_i \cap (M \setminus \mathrm{supp}\phi_i)$ we have $\phi_i(x) \cdot f_i(x) = 0 \cdot f_i(x) = \mathbf{0}$, so that f_i' is well defined. Now define

$$F : M \to \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$$

by

$$F(x) = \big(\phi_1(x), \dots, \phi_n(x), f_1'(x), \dots, f_n'(x)\big).$$

F is continuous, so by Lemma 12.9 we only need to verify that F is injective. Suppose $F(x) = F(y)$. Then $\phi_i(x) = \phi_i(y)$ and $f_i'(x) = f_i'(y)$ for all $i = 1, \dots, n$. Since $\sum_{i=1}^n \phi_i(x) = 1$, we have $\phi_i(x) = \phi_i(y) > 0$ for some i, so that $x, y \in U_i$. Then from $f_i'(x) = f_i'(y)$, we have $f_i(x) = f_i(y)$. But $f_i : U_i \to \mathbb{R}^m$ is injective, so that $x = y$. ◻